

Combinatorial tiling theory

Nico Van Cleemput

Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics, Computer Science and Statistics
Ghent University

25 September 2026



Work by

Andreas Dress, Daniel Huson, Olaf Delgado-Friedrichs,
Ludwig Balke, Matthew S. Delaney, Klaus Westphal, Rüdiger Zeller,
Rudolf Scharlau, Emil Molnár, Elizaveta Zamorzaeva, Zoran Lučić,
Milica Stojanović, Branko Grünbaum, Geoffrey C. Shephard,
Gunnar Brinkmann, Heidi Van den Camp, Pieter Goetschalckx,
Stan Schein, Kris Coolsaet



Andreas Dress



(1938 – 2024)

- 1 Introduction
 - Main characters
- 2 Delaney–Dress symbols
 - Construction
 - Definition
 - Working with Delaney–Dress symbols
- 3 Applications
 - Tile- k -transitive tilings
 - Isotoxal tilings
 - Azulenoids
- 4 Visualisation
 - Technique
 - Examples
- 5 Local symmetry-preserving operations
 - Introduction
 - Operations and Delaney–Dress symbols
 - Compression





Tilings

Definition

Tiling $\mathcal{T}$ = set of tiles $t_1, t_2, \dots$ with $t_i \subset \mathbb{E}^2$, t_i homeomorphic to $\overline{B}(0, 1)$, that satisfy the following conditions:

- 1 $\bigcup_{t \in \mathcal{T}} t = \mathbb{E}^2$
- 2 $\forall t_i, t_j (i \neq j) \in \mathcal{T} : t_i^\circ \cap t_j^\circ = \emptyset$ and $t_i \cap t_j$ is empty, point or line.
- 3 $\forall x \in \mathbb{E}^2 : x$ has a neighbourhood that only intersects a finite number of tiles.



Tilings

Definition

Tiling T = set of tiles $t_1, t_2, \dots$ with $t_i \subset \mathbb{E}^2$, $t_i \cap t_j = \emptyset$ and $t_i \cap t_j$ is empty, point or line. T covers $\overline{B}(0, 1)$, that satisfy the following conditions:

- 1 $\bigcup_{t \in T} t = \mathbb{E}^2$
- 2 $\forall t_i, t_j (i \neq j) \implies t_i \cap t_j = \emptyset$ and $t_i \cap t_j$ is empty, point or line.
- 3 $\forall x \in \mathbb{E}^2$ there is a neighbourhood that only intersects a finite number of tiles.

CENSORED – DO NOT READ

Tilings

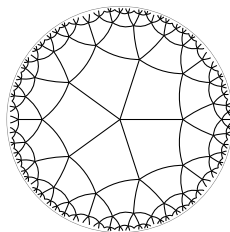
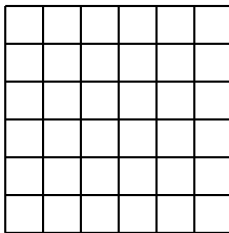
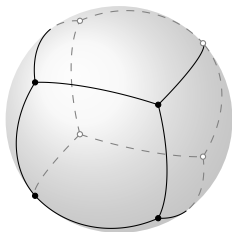


Tilings

- 1 Each tile is a closed topological disk.
- 2 Tiles cover the plane.
- 3 The intersection of any two distinct tiles is connected and has measure zero.
- 4 Uniformly bounded (tiles bounded above and below by fixed-radius disks).



Tilings



Tilings



Tilings



Tilings

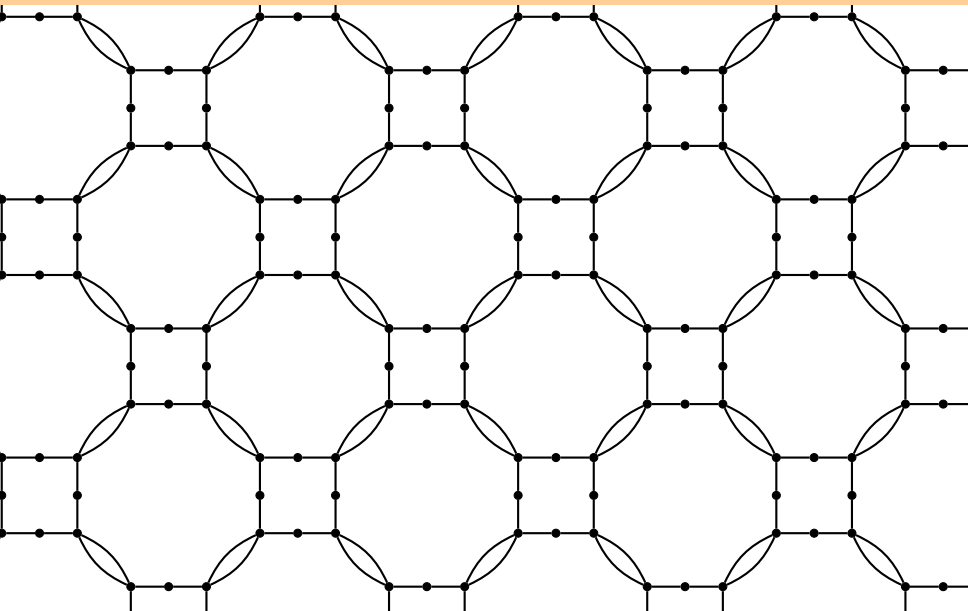
Periodic tiling

The whole configuration can be reproduced from a finite assembly of tiles by repeatedly shifting and copying in as many directions as needed.

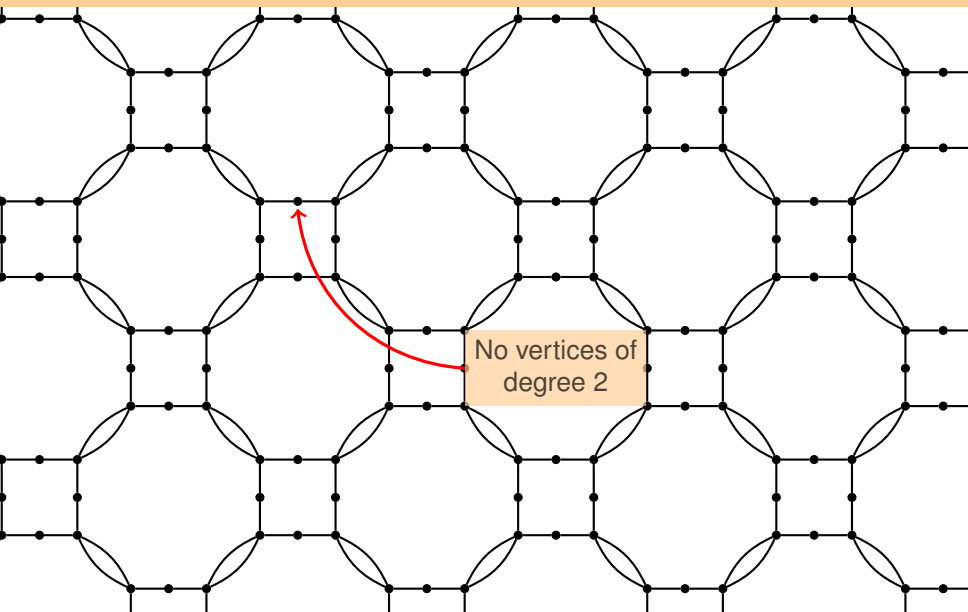
Euclidean plane

The symmetry group of the tiling contains at least two independent translations.

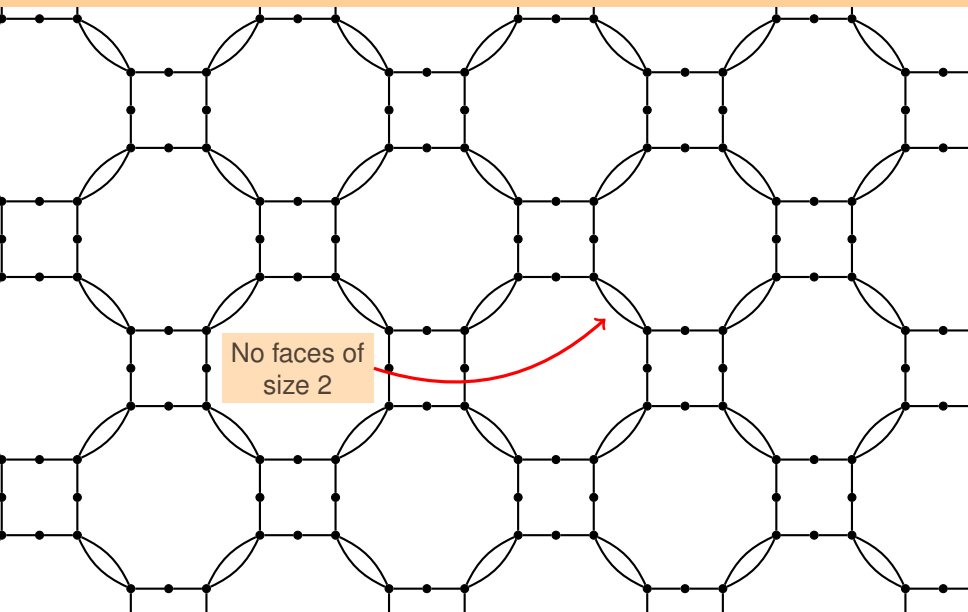
Tilings



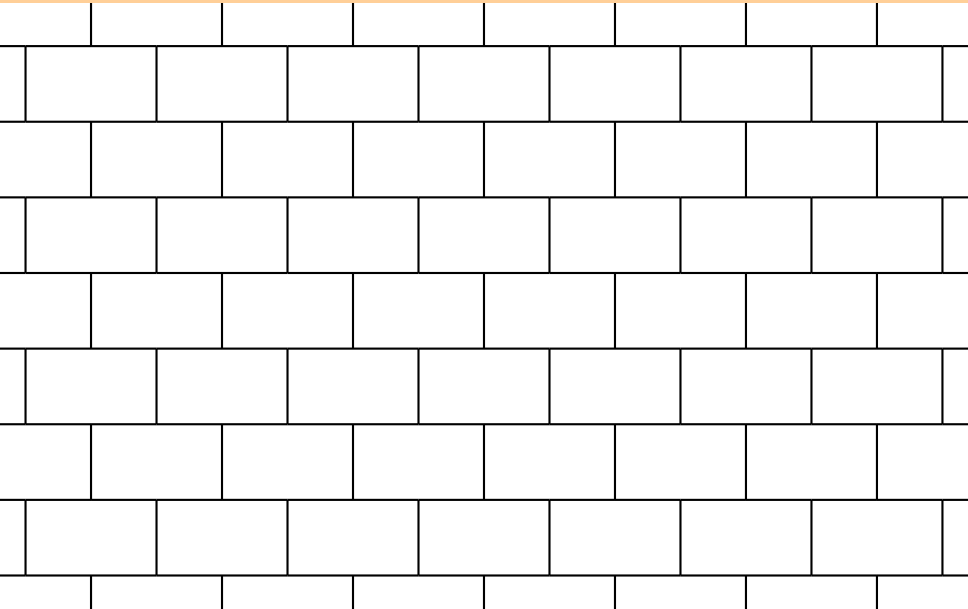
Tilings



Tilings



Tilings



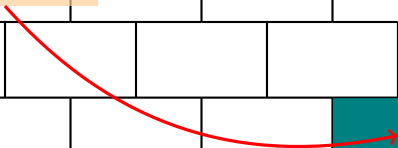
Tilings



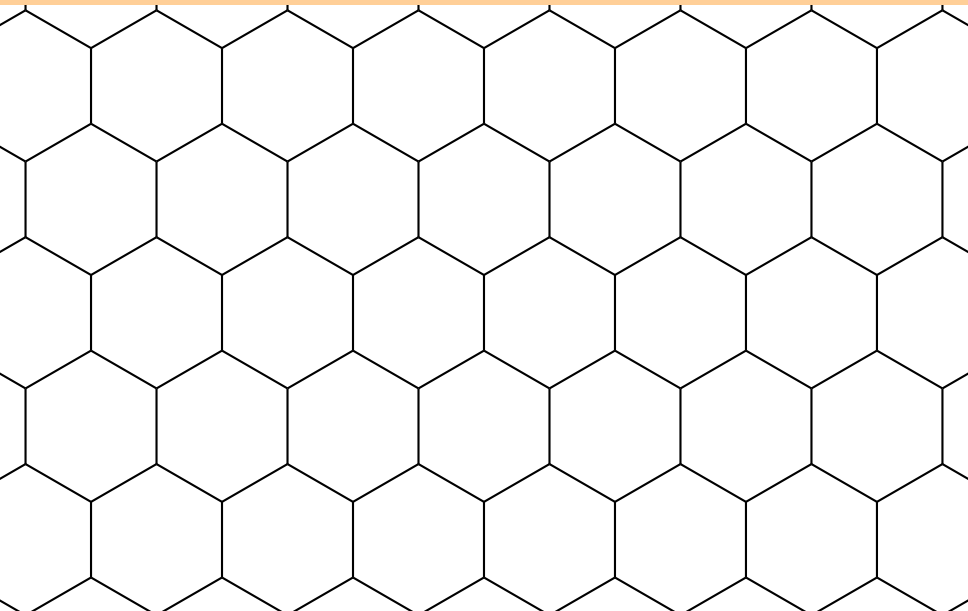
Vertex of degree 3

Tilings

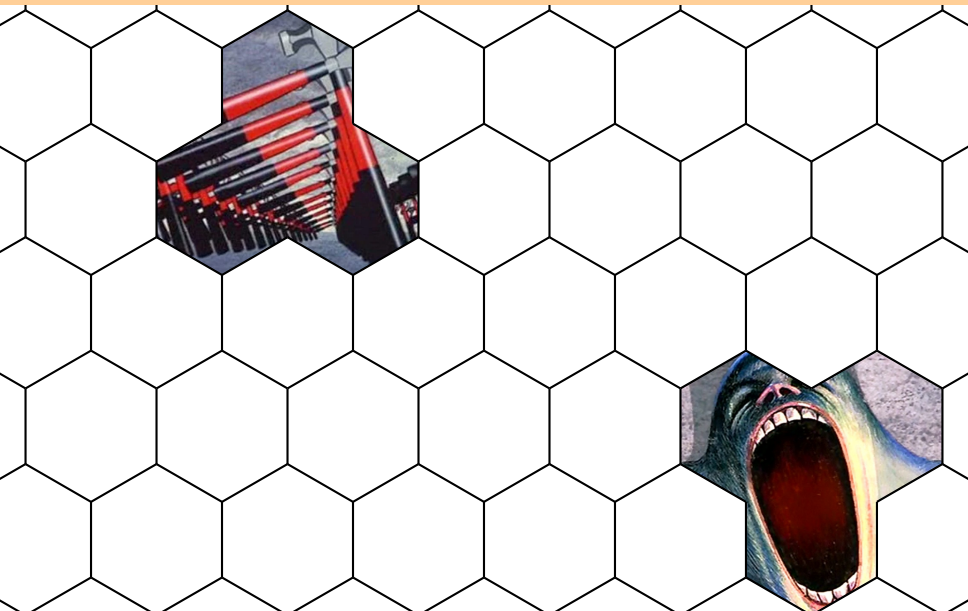
Hexagonal tile



Tilings



Tilings



Equivariant tilings

Definition

An *equivariant tiling* (T, Γ) is a tiling T together with a chosen symmetry group Γ .

- The symmetry group does not need to be the full symmetry group of the tiling.
- Canonical notion of “how many kinds of tiles, edges, vertices?”
- Symmetry-breaking is going from (T, Γ) to (T, Γ') with $\Gamma' \subset \Gamma$.
- Sometimes only (easily) drawable using markings.



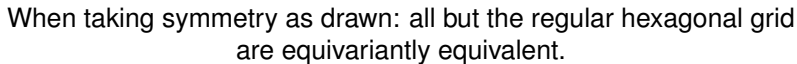
Equivariantly equivalent tilings

Definition

Two equivariant tilings (T, Γ) and (T', Γ') are *equivariantly equivalent* if

- 1 T and T' are topologically equivalent.
- 2 Γ and Γ' act “in the same way” on T and T' respectively.

There exists a homeomorphism Φ such that $\Phi(T) = T'$ and $\Gamma' = \Phi\Gamma\Phi^{-1}$.

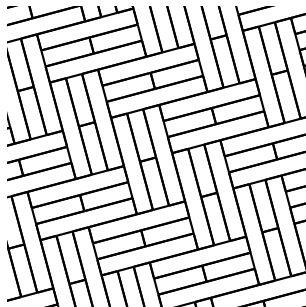
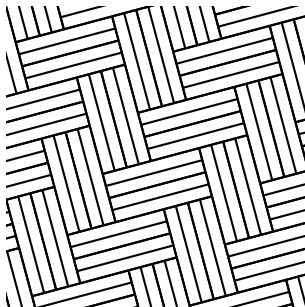


Collatz's question

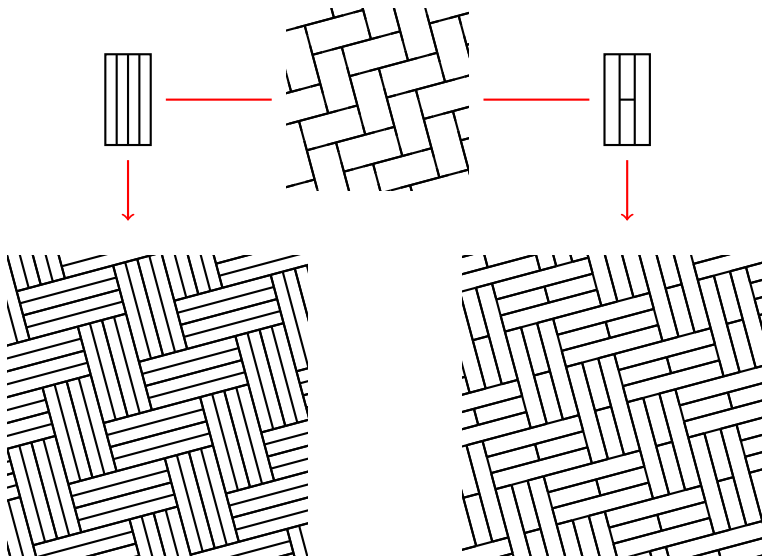
Question posed to the Bielefeld group by Lothar Collatz:

Question

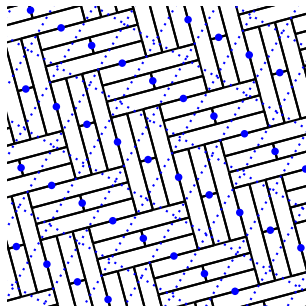
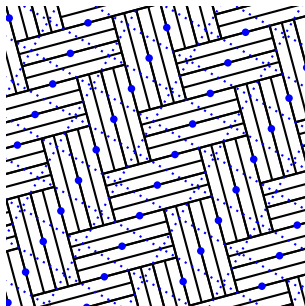
Are these two tilings equivariantly equivalent?



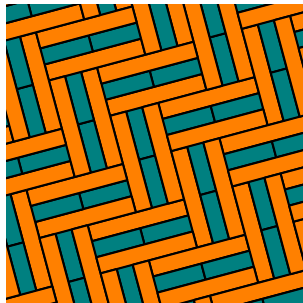
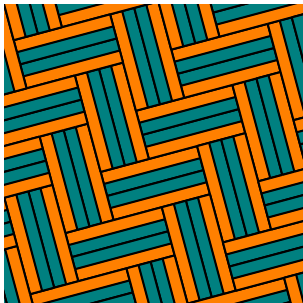
Collatz's question



Collatz's question



Collatz's question



Outline

- 1 Introduction
 - Main characters
- 2 Delaney–Dress symbols
 - Construction
 - Definition
 - Working with Delaney–Dress symbols
- 3 Applications
 - Tile- k -transitive tilings
 - Isotoxal tilings
 - Azulenoids
- 4 Visualisation
 - Technique
 - Examples
- 5 Local symmetry-preserving operations
 - Introduction
 - Operations and Delaney–Dress symbols
 - Compression



“It would be nice to be able to assert that the enumeration presented here is complete. Unfortunately we cannot do that, though it seems reasonable to suppose that the number of omissions (if any) must be quite small . . . Practically speaking, it seems that there is no way of doing this in a completely systematic manner.”

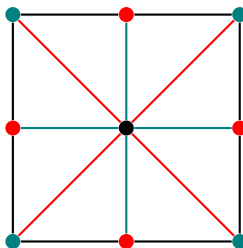
*Grünbaum, B., Löckenhoff, H. D., Shephard, G. C. and Temesvári, A.,
The Enumeration of Normal 2-Homeohedral Tilings,
Geom. Dedicata 19 (1985)*



Barycentric subdivision

- For each face: one point
- For each edge: one point
- For each vertex: one point

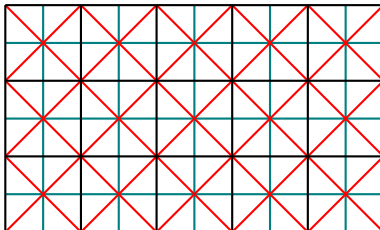
Incidence determines adjacency



Barycentric subdivision

- For each face: one point
- For each edge: **one point**
- For each vertex: **one point**

Incidence determines adjacency



Chamber system

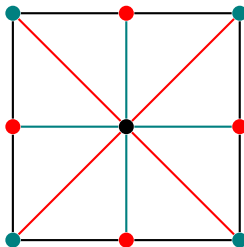
Define $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_i^2 = \mathbb{1} \rangle$

σ_0 : change the teal point (vertex).

σ_1 : change the red point (edge).

σ_2 : change the black point (face).

Chamber system $\mathcal{C}$ of $T =$ barycentric subdivision with Σ



Chamber system

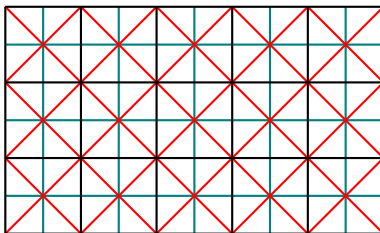
Define $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_i^2 = \mathbb{1} \rangle$

σ_0 : change the teal point (vertex).

σ_1 : change the red point (edge).

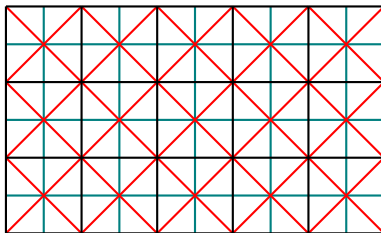
σ_2 : change the black point (face).

Chamber system $\mathcal{C}$ of $T =$ barycentric subdivision with Σ

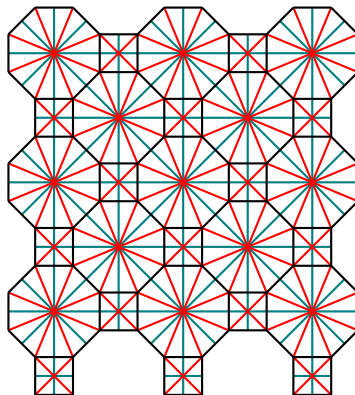
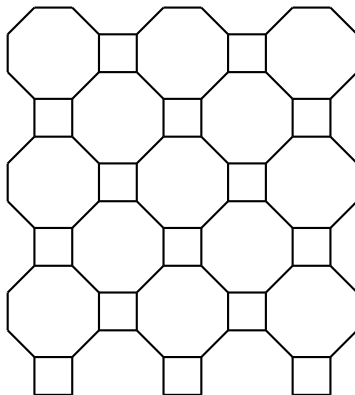


Compatibility

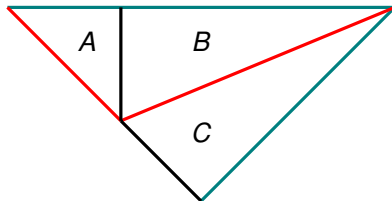
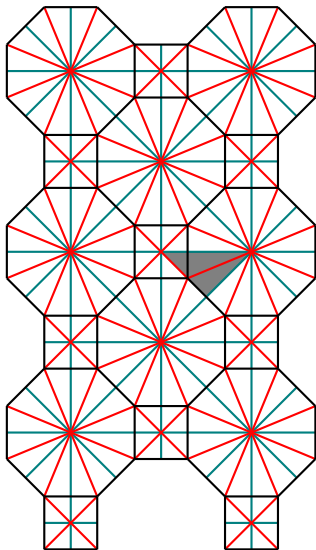
$$\gamma \cdot (\mathbf{c} \cdot \sigma_i) = (\gamma \cdot \mathbf{c}) \cdot \sigma_i$$



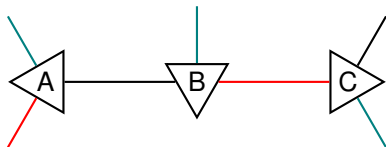
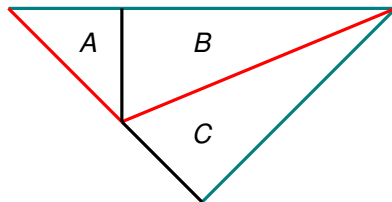
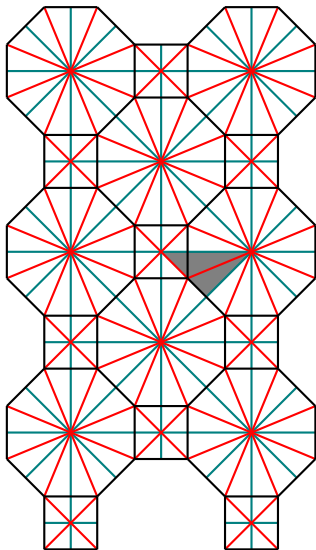
Delaney–Dress graph



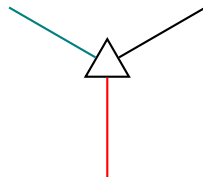
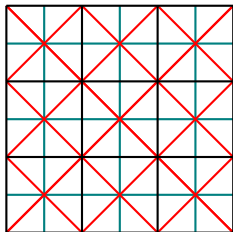
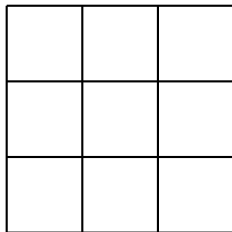
Delaney–Dress graph



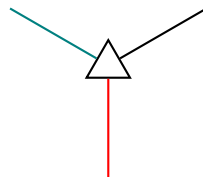
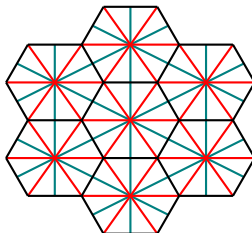
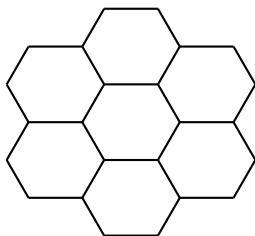
Delaney–Dress graph



Delaney–Dress graph



Delaney–Dress graph



Delaney–Dress graph

Different equivariant tilings can result in the same Delaney–Dress graph.

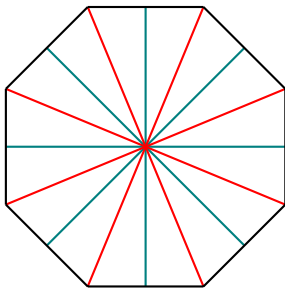


$\sigma_i \sigma_j$ -orbits

- $\sigma_i \sigma_i$ is always the identity

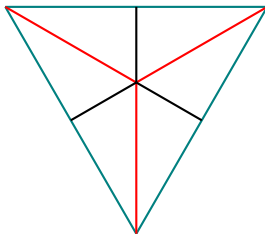
$\sigma_i \sigma_j$ -orbits

- $\sigma_i \sigma_i$ is always the identity
- $\sigma_0 \sigma_1$ corresponds to walking around a face



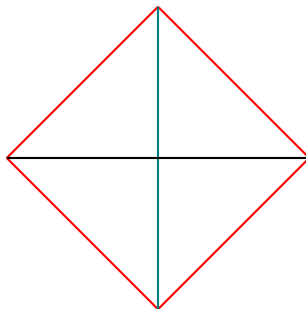
$\sigma_i \sigma_j$ -orbits

- $\sigma_i \sigma_i$ is always the identity
- $\sigma_0 \sigma_1$ corresponds to walking around a face
- $\sigma_1 \sigma_2$ corresponds to walking around a vertex



$\sigma_i \sigma_j$ -orbits

- $\sigma_i \sigma_i$ is always the identity
- $\sigma_0 \sigma_1$ corresponds to walking around a face
- $\sigma_1 \sigma_2$ corresponds to walking around a vertex
- $\sigma_0 \sigma_2$ corresponds to walking around an edge



$\sigma_i\sigma_j$ -orbits

Define functions $m_{ij} : \mathcal{C} \rightarrow \mathbb{N}; c \mapsto m_{ij}(c)$ with $m_{ij}(c)$ the smallest positive integer such that $c(\sigma_i\sigma_j)^{m_{ij}(c)} = c$.

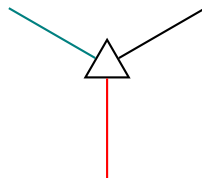
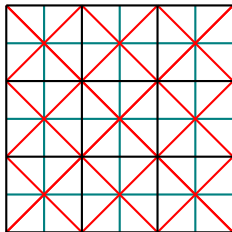
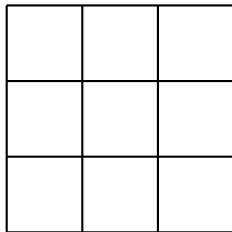
- $m_{01}(c)$ is the size of the face corresponding to the chamber c .
- $m_{12}(c)$ is the degree of the vertex corresponding to the chamber c .
- $m_{02}(c)$ is always two.
- For higher dimensions: m_{ij} is always two when $|i - j| > 1$.
- m_{ij} is constant on Γ -orbits $\Rightarrow$ can be carried over to the Delaney–Dress graph.



Delaney–Dress symbol

The Delaney–Dress symbol of a periodic tiling is $(\mathcal{D}; m_{01}, m_{12})$.

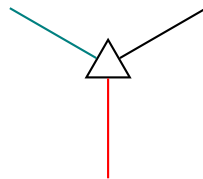
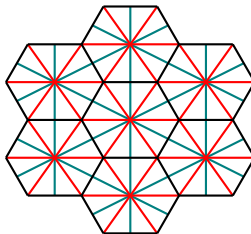
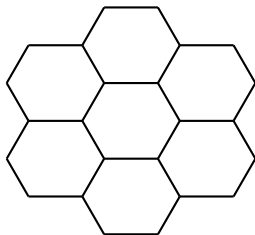
Delaney–Dress symbol



$$m_{01}(c) = 4$$

$$m_{12}(c) = 4$$

Delaney–Dress symbol



$$m_{01}(c) = 6$$

$$m_{12}(c) = 3$$

Delaney–Dress symbol

Theorem (Dress, 1985)

Two equivariant tilings are equivariantly equivalent iff their respective Delaney–Dress symbols are isomorphic.



$\sigma_i\sigma_j$ -orbits (revisited)

Define functions $r_{ij} : \mathcal{D} \rightarrow \mathbb{N}$; $d \mapsto r_{ij}(d)$ with $r_{ij}(d)$ the smallest number for which $d(\sigma_i\sigma_j)^{r_{ij}(d)} = d$.

- $m_{ij}(d)$ is a multiple of $r_{ij}(d)$.
- $v_{ij} : \mathcal{D} \rightarrow \mathbb{N}$; $d \mapsto m_{ij}(d)/r_{ij}(d)$ can be defined and is constant on $\sigma_i\sigma_j$ -orbits.
- $v_{ij}(d)$ is the **branching number** of the orbit.
- $v_{ij}(d)$ is the order of the rotation in Γ about the vertex/edge/face-centre.



Theorem

$(\mathcal{D}; m_{01}, m_{12})$ is the Delaney–Dress symbol of a periodic tiling iff

- 1 $\mathcal{D}$ is finite
- 2 Σ acts transitively on $\mathcal{D}$
- 3 m_{01} is constant on $\langle \sigma_0, \sigma_1 \rangle$ -orbits and $\forall d \in \mathcal{D} : d(\sigma_0 \sigma_1)^{m_{01}(d)} = d$
- 4 m_{12} is constant on $\langle \sigma_1, \sigma_2 \rangle$ -orbits and $\forall d \in \mathcal{D} : d(\sigma_1 \sigma_2)^{m_{12}(d)} = d$
- 5 $\forall d \in \mathcal{D} : d(\sigma_0 \sigma_2)^2 = d$
- 6 Curvature condition

Curvature condition

$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right)$$

- $K(\mathcal{D}) < 0 \rightarrow$ hyperbolic plane
- $K(\mathcal{D}) = 0 \rightarrow$ euclidean plane
- $K(\mathcal{D}) > 0 \rightarrow$ sphere iff $\forall d \in \mathcal{D} : \frac{4}{v_{ij}(d)K(\mathcal{D})} \in \mathbb{N}$

Spherical case

$d \in \mathcal{D}$ is a spherical triangle.

$$\begin{aligned}\text{area}(d) &= \left(\frac{\pi}{m_{01}(d)} + \frac{\pi}{m_{12}(d)} + \frac{\pi}{2} \right) - \pi \\ &= \pi \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right)\end{aligned}$$

Spherical case

Summing over the whole symbol

$$\begin{aligned}\sum_{d \in \mathcal{D}} \text{area}(d) &= \pi K(\mathcal{D}) \\ &= \frac{\text{area}(S^2)}{|\Gamma|} = \frac{4\pi}{|\Gamma|}\end{aligned}$$

$$\Rightarrow |\Gamma| = \frac{4}{K(\mathcal{D})}$$

Spherical case

For each $d \in \mathcal{D}$: the pair $\{i, j\}$ picks a point of the tiling: a vertex, an edge-midpoint, or a face-centre.

$v_{ij}(d)$ is the order of the rotation stabilising that point.

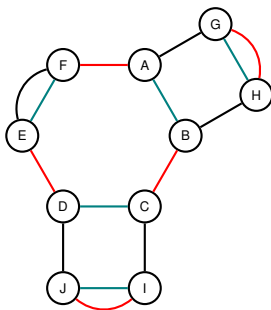
$$\Rightarrow v_{ij}(d) \mid \frac{4}{K(\mathcal{D})} \Rightarrow \frac{4}{v_{ij}(d)K(\mathcal{D})} \in \mathbb{N}$$



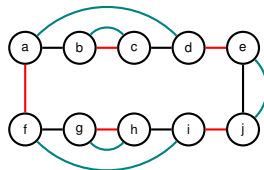
What the symbol tells you

- Orbits of vertices, edges, and faces
 - Count
 - Type (degree/size)
 - Vertex/edge/face figure
 - Colourability
- Chirality vs. achirality (presence of orientation-reversing symmetries)
- Duality (dual and self-dual tilings)
- Orbifold notation
- ...

Isomorphism

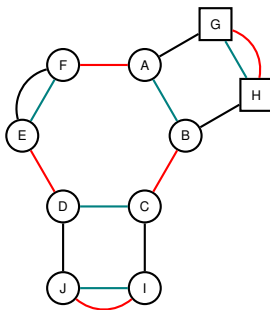


	m_{01}	m_{12}
A	3	5
B	3	5
C	3	5
D	3	5
E	3	5
F	3	5
G	6	5
H	6	5
I	3	5
J	3	5

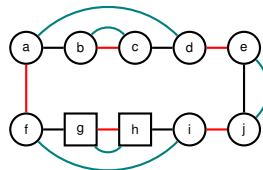


	m_{01}	m_{12}
a	3	5
b	3	5
c	3	5
d	3	5
e	3	5
f	3	5
g	6	5
h	6	5
i	3	5
j	3	5

Isomorphism

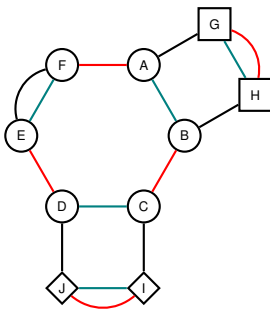


	m_{01}	m_{12}
A	3	5
B	3	5
C	3	5
D	3	5
E	3	5
F	3	5
G	6	5
H	6	5
I	3	5
J	3	5

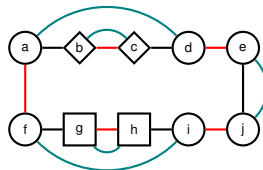


	m_{01}	m_{12}
a	3	5
b	3	5
c	3	5
d	3	5
e	3	5
f	3	5
g	6	5
h	6	5
i	3	5
j	3	5

Isomorphism

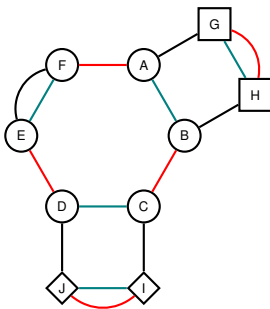


	m_{01}	m_{12}
A	3	5
B	3	5
C	3	5
D	3	5
E	3	5
F	3	5
G	6	5
H	6	5
I	3	5
J	3	5

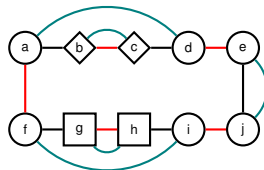


	m_{01}	m_{12}
a	3	5
b	3	5
c	3	5
d	3	5
e	3	5
f	3	5
g	6	5
h	6	5
i	3	5
j	3	5

Isomorphism

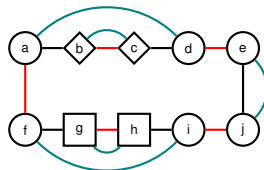
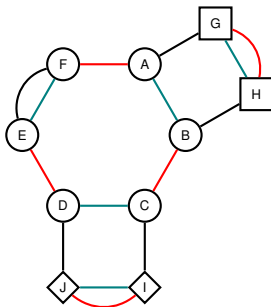


	m_{01}	m_{12}
A	3	5
B	3	5
C	3	5
D	3	5
E	3	5
F	3	5
G	6	5
H	6	5
I	3	5
J	3	5

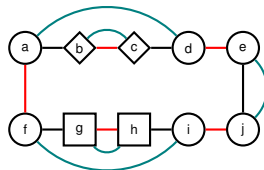
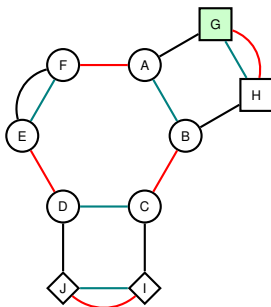


	m_{01}	m_{12}
a	3	5
b	3	5
c	3	5
d	3	5
e	3	5
f	3	5
g	6	5
h	6	5
i	3	5
j	3	5

Isomorphism

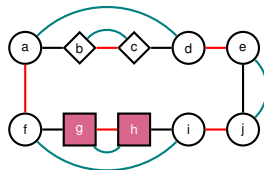
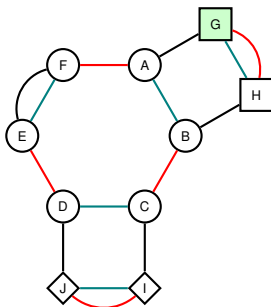


Isomorphism

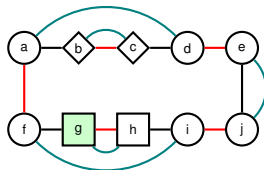


A	↦
B	↦
C	↦
D	↦
E	↦
F	↦
G	↦
H	↦
I	↦
J	↦

Isomorphism

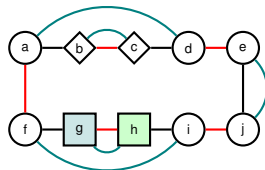
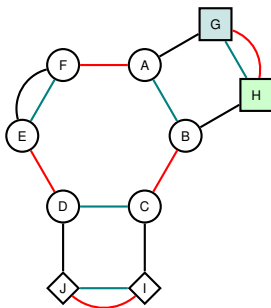


A	$\mapsto$
B	$\mapsto$
C	$\mapsto$
D	$\mapsto$
E	$\mapsto$
F	$\mapsto$
G	$\mapsto$
H	$\mapsto$
I	$\mapsto$
J	$\mapsto$



A	$\Rightarrow$	
B	$\Rightarrow$	
C	$\Rightarrow$	
D	$\Rightarrow$	
E	$\Rightarrow$	
F	$\Rightarrow$	
G	$\Rightarrow$	g
H	$\Rightarrow$	
I	$\Rightarrow$	
J	$\Rightarrow$	

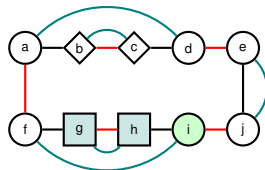
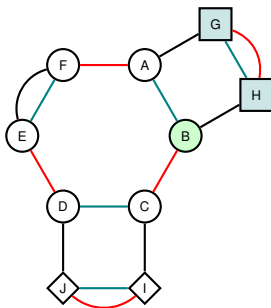
Isomorphism



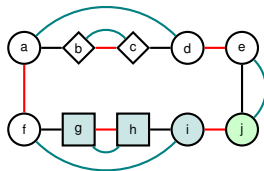
A	→
B	→
C	→
D	→
E	→
F	→
G	→
H	→
I	→
J	→

g
h

Isomorphism

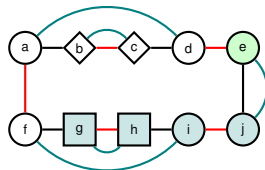
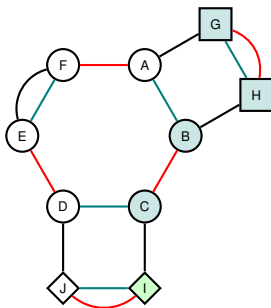


A	→	
B	→	i
C	→	
D	→	
E	→	
F	→	
G	→	g
H	→	h
I	→	
J	→	



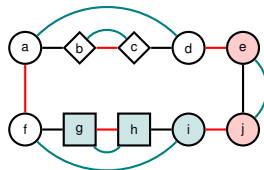
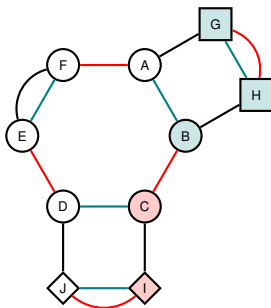
A	→	
B	→	i
C	→	j
D	→	
E	→	
F	→	
G	→	g
H	→	h
I	→	
J	→	

Isomorphism



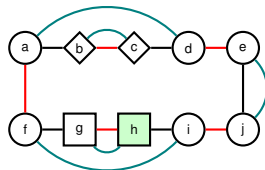
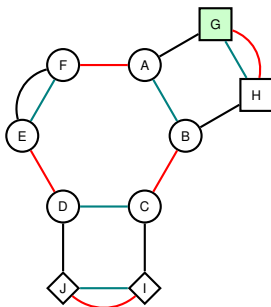
A	$\mapsto$	
B	$\mapsto$	i
C	$\mapsto$	j
D	$\mapsto$	
E	$\mapsto$	
F	$\mapsto$	
G	$\mapsto$	g
H	$\mapsto$	h
I	$\mapsto$	e
J	$\mapsto$	

Isomorphism



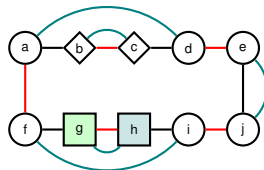
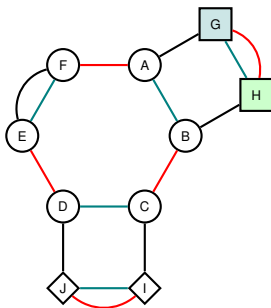
A	↦	
B	↦	i
C	↦	j
D	↦	
E	↦	
F	↦	
G	↦	g
H	↦	h
I	↦	e
J	↦	

Isomorphism



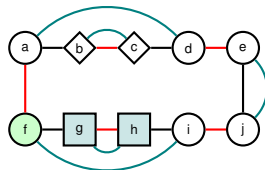
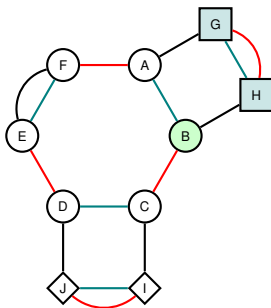
A	↦	
B	↦	
C	↦	
D	↦	
E	↦	
F	↦	
G	↦	
H	↦	h
I	↦	
J	↦	

Isomorphism



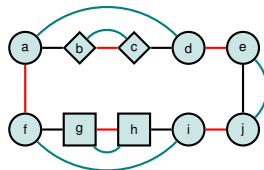
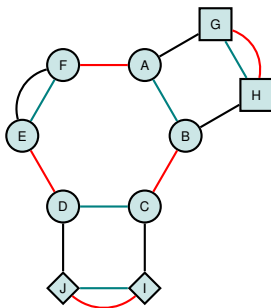
A	→	
B	→	
C	→	
D	→	
E	→	
F	→	
G	→	h
H	→	g
I	→	
J	→	

Isomorphism



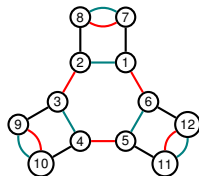
A	$\mapsto$	
B	$\mapsto$	f
C	$\mapsto$	
D	$\mapsto$	
E	$\mapsto$	
F	$\mapsto$	
G	$\mapsto$	h
H	$\mapsto$	g
I	$\mapsto$	
J	$\mapsto$	

Isomorphism

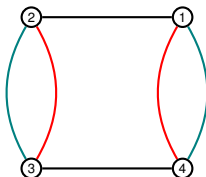


A	$\mapsto$	i
B	$\mapsto$	f
C	$\mapsto$	a
D	$\mapsto$	d
E	$\mapsto$	e
F	$\mapsto$	j
G	$\mapsto$	h
H	$\mapsto$	g
I	$\mapsto$	b
J	$\mapsto$	c

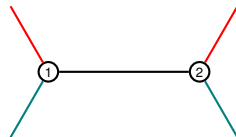
Maximal symbol



	m_{01}	m_{12}
1	6	4
2	6	4
3	6	4
4	6	4
5	6	4
6	6	4
7	3	4
8	3	4
9	3	4
10	3	4
11	3	4
12	3	4

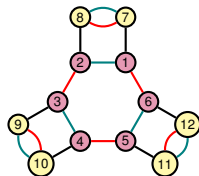


	m_{01}	m_{12}
1	6	4
2	3	4
3	3	4
4	6	4

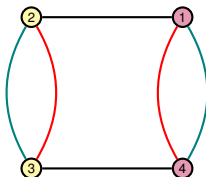


	m_{01}	m_{12}
1	6	4
2	3	4

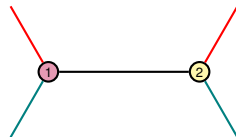
Maximal symbol



	m_{01}	m_{12}
1	6	4
2	6	4
3	6	4
4	6	4
5	6	4
6	6	4
7	3	4
8	3	4
9	3	4
10	3	4
11	3	4
12	3	4



	m_{01}	m_{12}
1	6	4
2	3	4
3	3	4
4	6	4



	m_{01}	m_{12}
1	6	4
2	3	4

Maximal symbol

- If there is a map from a symbol $\mathcal{D}$ to a smaller symbol $\mathcal{D}'$, then $\mathcal{D}$ is a symmetry-breaking of $\mathcal{D}'$.
- If there is no map from a symbol $\mathcal{D}$ to a strictly smaller symbol $\mathcal{D}'$, then $\mathcal{D}$ is maximal.
- A maximal symbol has maximal symmetry.
- Every symbol has a maximal symbol (possibly itself).

Theorem

Two equivariant tilings are topologically equivalent iff their maximal Delaney–Dress symbols are isomorphic.

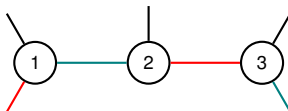


Delaney–Dress symbols per Delaney–Dress graph

Given a Delaney–Dress graph, the number of compatible Delaney–Dress symbols is:

- a finite number for Euclidean tilings
- an infinite number for hyperbolic tilings
- possibly an infinite number for spherical tilings

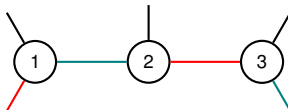




	m_{01}	m_{12}
1	3	p
2	3	4
3	3	4

$$\begin{aligned}
 K(\mathcal{D}) &= \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) \\
 &= 3 \cdot \frac{1}{3} + \frac{1}{p} + 2 \cdot \frac{1}{4} - 3 \cdot \frac{1}{2} \\
 &= \frac{1}{p}
 \end{aligned}$$

$\Rightarrow$ possibly spherical



	m_{01}	m_{12}
1	3	p
2	3	4
3	3	4

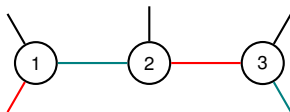
	r_{01}	r_{12}
1	3	1
2	3	2
3	3	2

	v_{01}	v_{12}
1	1	p
2	1	2
3	1	2

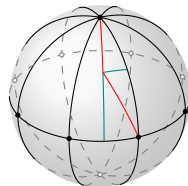
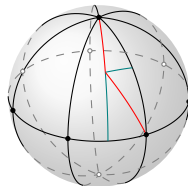
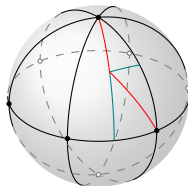
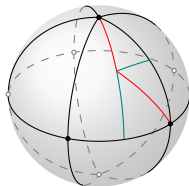
	$\frac{4}{v_{01}K(\mathcal{D})}$	$\frac{4}{v_{12}K(\mathcal{D})}$
1	$4p$	4
2	$4p$	$2p$
3	$4p$	$2p$

	$\frac{4}{v_{01}K(\mathcal{D})}$	$\frac{4}{v_{12}K(\mathcal{D})}$
1	$4p$	4
2	$4p$	$2p$
3	$4p$	$2p$

All are integers, so for any value of p , the corresponding Delaney–Dress symbol is a valid spherical symbol.



	m_{01}	m_{12}
1	3	p
2	3	4
3	3	4



Geometry minimal

A Delaney–Dress symbol is **geometry minimal** if it is

- spherical and the branching values v_{ij} are at most 5.
- Euclidean (always geometry minimal).
- hyperbolic and no branching value can be lowered without pushing the curvature to zero or positive values.

When restricted to geometry-minimal symbols, there are only finitely many Delaney–Dress symbols for each Delaney–Dress graph.



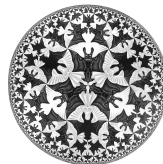
Outline

- 1 Introduction
 - Main characters
- 2 Delaney–Dress symbols
 - Construction
 - Definition
 - Working with Delaney–Dress symbols
- 3 Applications
 - Tile- k -transitive tilings
 - Isotoxal tilings
 - Azulenoids
- 4 Visualisation
 - Technique
 - Examples
- 5 Local symmetry-preserving operations
 - Introduction
 - Operations and Delaney–Dress symbols
 - Compression



Enumeration

- Tile- k -transitive tilings of the plane:
 - Tile-transitive tilings: 93 equivariant tilings (11 topological types)
 - Tile-2-transitive tilings: 1,270 equivariant tilings (508 topological types)
 - Tile-3-transitive tilings: 48,231 equivariant tilings (16,774 topological types)
 - 4-colourable tile-4-transitive tilings: 4,044 topological types
- Isotoxal tilings:
 - 14 infinite families across all three geometries
 - 30 equivariant tilings of the plane
- Minimal non-transitive tilings of the plane: 37 equivariant tilings
- Heaven-and-hell tilings of the plane: 23 topological types

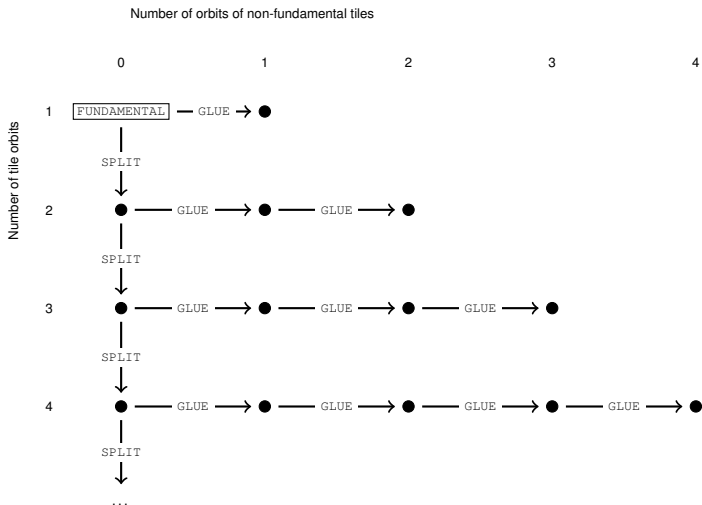


Tile- k -transitive tilings

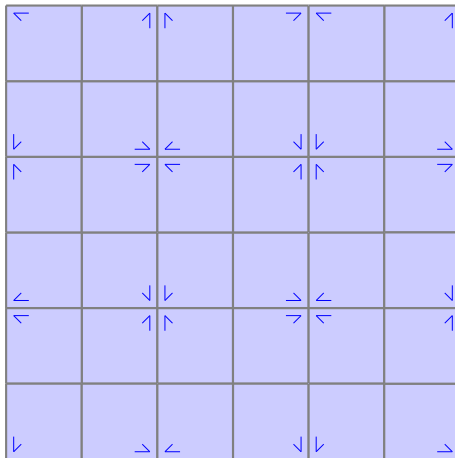
- Basic idea:
 - Every fundamental tile- k -transitive tiling can be constructed from a fundamental tile- $(k - 1)$ -transitive tiling **by splitting tiles from one tile orbit.**
 - Every tile- k -transitive tiling with $0 < k' \leq k$ non-fundamental tiles can be obtained by **gluing together one type of fundamental tiles** of a tile- k -transitive tiling with $(k' - 1)$ non-fundamental tiles.
- Needs fundamental tile-transitive tilings as input
- Using Delaney–Dress symbols this can be performed very mechanically
- However, there are several special cases to consider



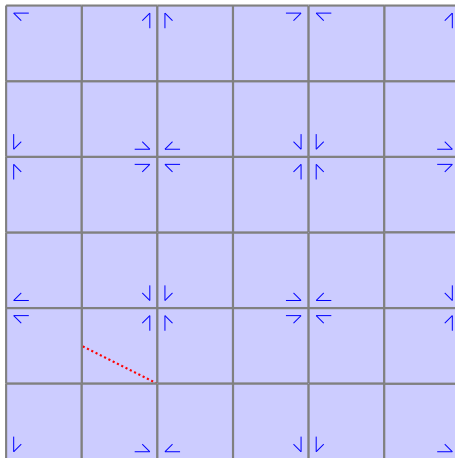
Tile- k -transitive tilings



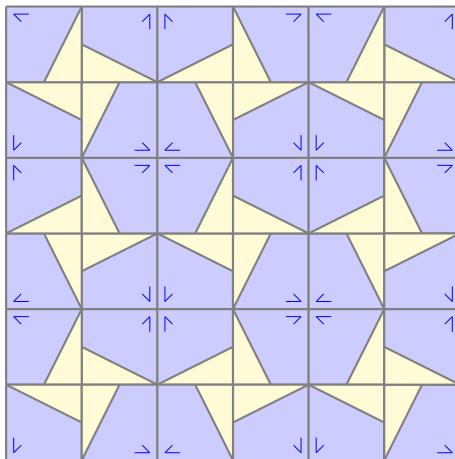
Tile- k -transitive tilings



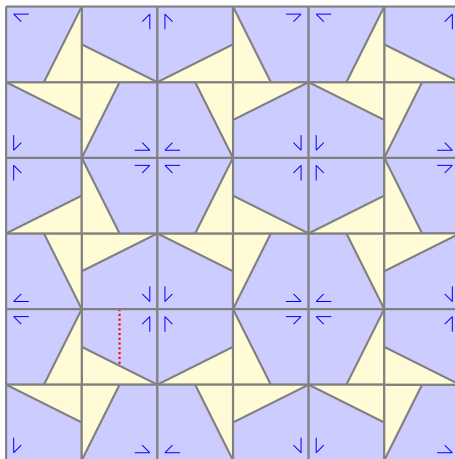
Tile- k -transitive tilings



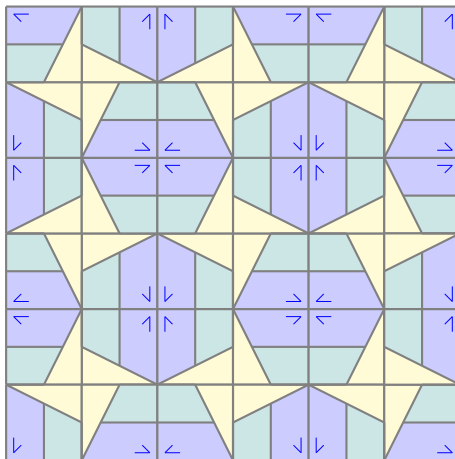
Tile- k -transitive tilings



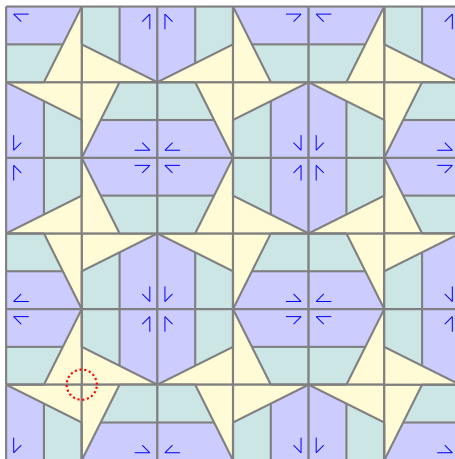
Tile- k -transitive tilings



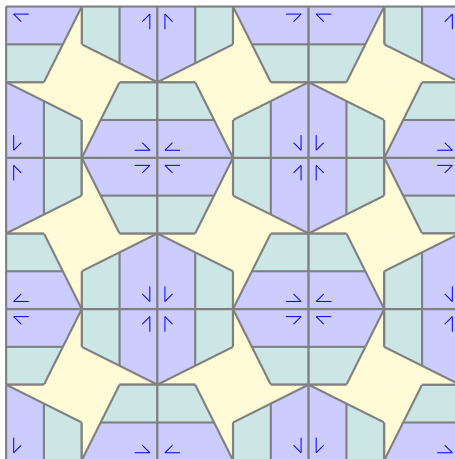
Tile- k -transitive tilings



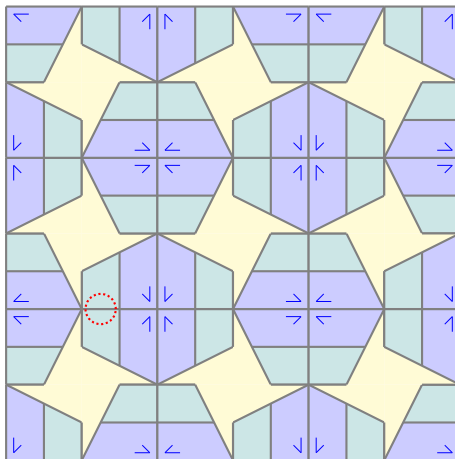
Tile- k -transitive tilings



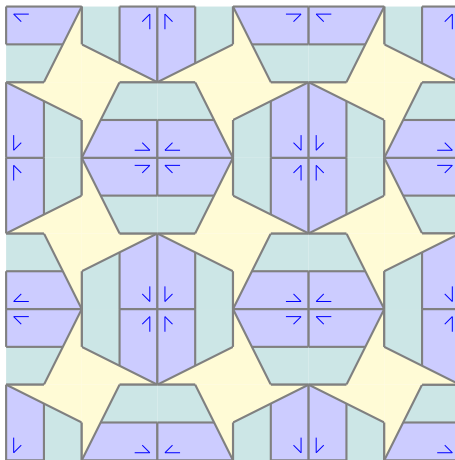
Tile- k -transitive tilings



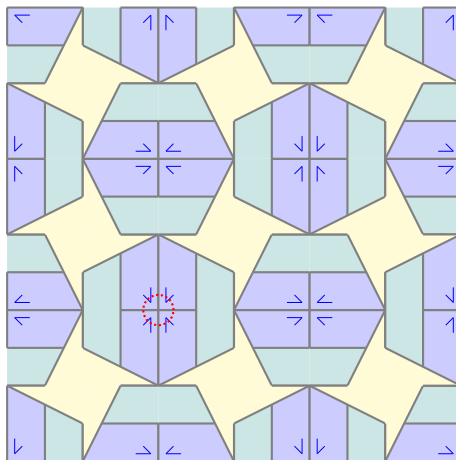
Tile- k -transitive tilings



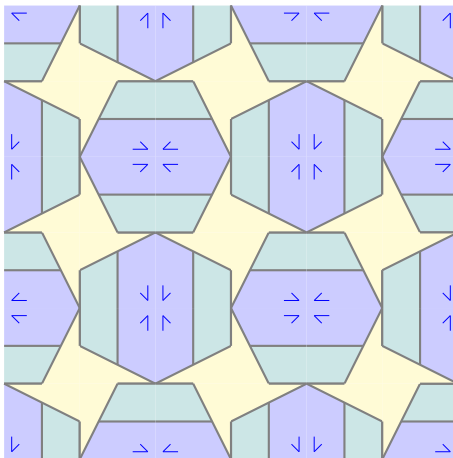
Tile- k -transitive tilings



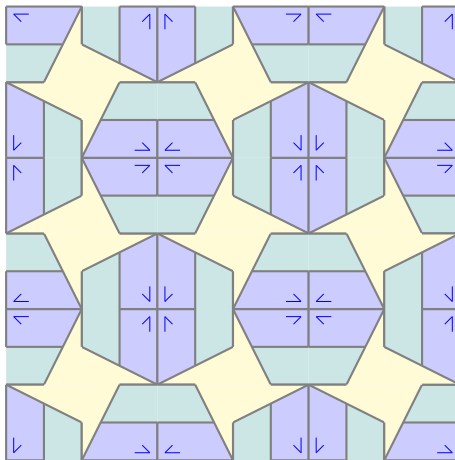
Tile- k -transitive tilings



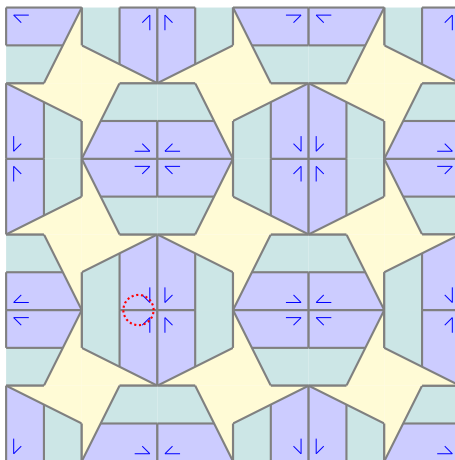
Tile- k -transitive tilings



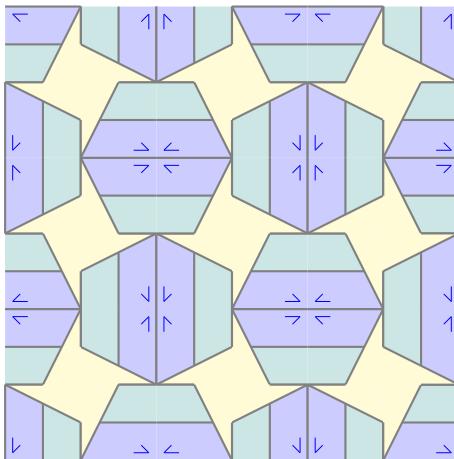
Tile- k -transitive tilings



Tile- k -transitive tilings



Tile- k -transitive tilings



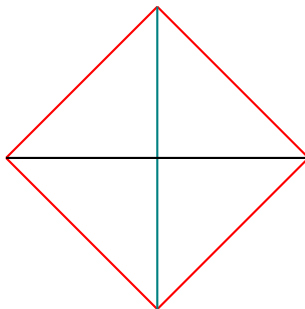
Isotoxal tilings

- A tiling is isotoxal if the symmetry group acts transitively on the edges.



Isotoxal tilings

- A tiling is isotoxal if the symmetry group acts transitively on the edges.
- In the chamber system, an edge corresponds to a $\sigma_0\sigma_2$ -4-cycle.



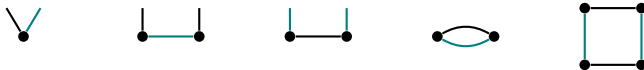
Isotoxal tilings

- A tiling is isotoxal if the symmetry group acts transitively on the edges.
- In the chamber system, an edge corresponds to a $\sigma_0\sigma_2$ -4-cycle.
- In the Delaney–Dress symbol, this 4-cycle becomes a quotient of a 4-cycle.

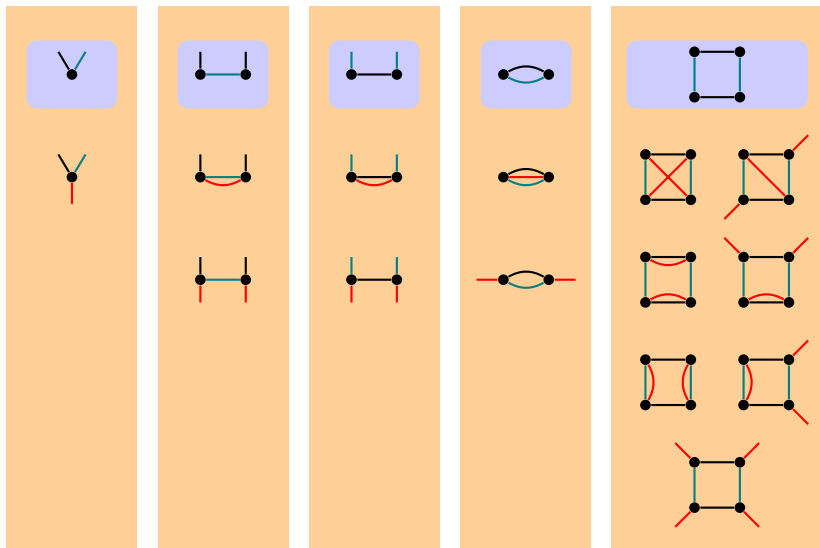


Isotoxal tilings

- A tiling is isotoxal if the symmetry group acts transitively on the edges.
- In the chamber system, an edge corresponds to a $\sigma_0\sigma_2$ -4-cycle.
- In the Delaney–Dress symbol, this 4-cycle becomes a quotient of a 4-cycle.
- This is the only $\sigma_0\sigma_2$ -component and it contains all vertices.



Isotoxal tilings



Isotoxal tilings



$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

Isotoxal tilings



$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$\Rightarrow \frac{1}{m_{01}} + \frac{1}{m_{12}} - \frac{1}{2} = 0$$

Isotoxal tilings



$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$\Rightarrow \frac{1}{m_{01}} + \frac{1}{m_{12}} - \frac{1}{2} = 0 \Rightarrow m_{01} = \frac{2m_{12}}{m_{12} - 2} \text{ and } m_{12} = \frac{2m_{01}}{m_{01} - 2}$$

Isotoxal tilings



$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$\Rightarrow \frac{1}{m_{01}} + \frac{1}{m_{12}} - \frac{1}{2} = 0 \Rightarrow m_{01} = \frac{2m_{12}}{m_{12} - 2} \text{ and } m_{12} = \frac{2m_{01}}{m_{01} - 2}$$

$$m_{01} \geq 3 \text{ and } m_{12} \geq 3, \text{ so } m_{01} \leq 6 \text{ and } m_{12} \leq 6$$

Isotoxal tilings



$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

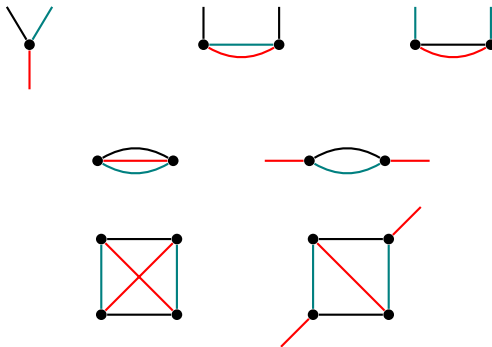
$$\Rightarrow \frac{1}{m_{01}} + \frac{1}{m_{12}} - \frac{1}{2} = 0 \Rightarrow m_{01} = \frac{2m_{12}}{m_{12} - 2} \text{ and } m_{12} = \frac{2m_{01}}{m_{01} - 2}$$

$m_{01} \geq 3$ and $m_{12} \geq 3$, so $m_{01} \leq 6$ and $m_{12} \leq 6$

$$(m_{01}, m_{12}) \in \{(3, 6), (4, 4), (6, 3)\}$$

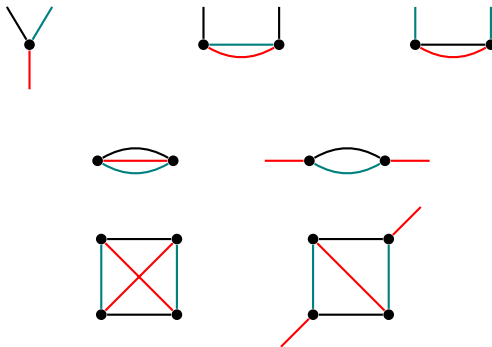
Isotoxal tilings

- $(m_{01}, m_{12}) \in \{(3, 6), (4, 4), (6, 3)\}$ can be used for all symbols with only one $\sigma_0\sigma_1$ - and only one $\sigma_1\sigma_2$ -orbit.



Isotoxal tilings

- $(m_{01}, m_{12}) \in \{(3, 6), (4, 4), (6, 3)\}$ can be used for all symbols with only one $\sigma_0\sigma_1$ - and only one $\sigma_1\sigma_2$ -orbit.
- With the additional restriction that $\forall d \in \mathcal{D} : r_{ij}(d) \mid m_{ij}(d)$.



Isotoxal tilings

- The remaining graphs have two or four vertices each.
- Each graph has at most two $\sigma_0\sigma_1$ - and at most two $\sigma_1\sigma_2$ -orbits.

$$|\mathcal{D}| = 2$$

$$K(\mathcal{D}) = \frac{1}{m_{01}(d_1)} + \frac{1}{m_{12}(d_1)} - \frac{1}{2} \\ + \frac{1}{m_{01}(d_2)} + \frac{1}{m_{12}(d_2)} - \frac{1}{2}$$

$$K(\mathcal{D}) = \frac{1}{m_{01}} + \frac{1}{m'_{01}} + \frac{1}{m_{12}} + \frac{1}{m'_{12}} - 1$$

$$|\mathcal{D}| = 4$$

$$K(\mathcal{D}) = \frac{1}{m_{01}(d_1)} + \frac{1}{m_{12}(d_1)} - \frac{1}{2} \\ + \frac{1}{m_{01}(d_2)} + \frac{1}{m_{12}(d_2)} - \frac{1}{2} \\ + \frac{1}{m_{01}(d_3)} + \frac{1}{m_{12}(d_3)} - \frac{1}{2} \\ + \frac{1}{m_{01}(d_4)} + \frac{1}{m_{12}(d_4)} - \frac{1}{2}$$

$$K(\mathcal{D}) = 2 \left(\frac{1}{m_{01}} + \frac{1}{m'_{01}} + \frac{1}{m_{12}} + \frac{1}{m'_{12}} - 1 \right)$$

Isotoxal tilings

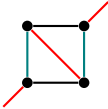
Solve $\frac{1}{k} + \frac{1}{l} + \frac{1}{n} + \frac{1}{m} - 1 = 0$ with $3 \leq k \leq l \leq m \leq n$.

k	l	m	n
3	3	4	12
3	3	6	6
3	4	4	6
4	4	4	4

Isotoxal tilings

D-graph	Orbits	r_{ij}	Solutions
	(m_{01}, m_{12})	$(1, 1)$	$(3, 6), (4, 4), (6, 3)$
	(m_{01}, m_{12})	$(1, 2)$	$(3, 6), (4, 4)$
	(m_{01}, m_{12})	$(2, 1)$	$(4, 4), (6, 3)$

Isotoxal tilings

D-graph	Orbits	r_{ij}	Solutions
	(m_{01}, m_{12})	$(1, 1)$	$(3, 6), (4, 4), (6, 3)$
	(m_{01}, m_{12})	$(2, 2)$	$(4, 4)$
	(m_{01}, m_{12})	$(2, 2)$	$(4, 4)$
	(m_{01}, m_{12})	$(4, 4)$	$(4, 4)$

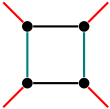
Isotoxal tilings

D-graph	Orbits	r_{ij}	Solutions
	$(m_{01}, m_{12}, m'_{12})$	$(2, 1, 1)$	$(4, 3, 6), (4, 4, 4), (6, 3, 3)$
	$(m_{01}, m'_{01}, m_{12})$	$(1, 1, 2)$	$(3, 3, 6), (3, 6, 4), (4, 4, 4)$

Isotoxal tilings

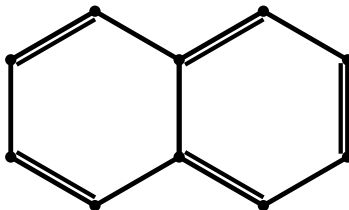
D-graph	Orbits	r_{ij}	Solutions
	$(m_{01}, m_{12}, m'_{12})$	$(2, 1, 1)$	$(4, 3, 6), (4, 4, 4), (6, 3, 3)$
	$(m_{01}, m'_{01}, m_{12})$	$(1, 1, 2)$	$(3, 3, 6), (3, 6, 4), (4, 4, 4)$
	$(m_{01}, m_{12}, m'_{12})$	$(4, 2, 1)$	$(4, 4, 4), (4, 6, 3)$
	$(m_{01}, m'_{01}, m_{12})$	$(2, 1, 4)$	$(4, 4, 4), (6, 3, 4)$

Isotoxal tilings

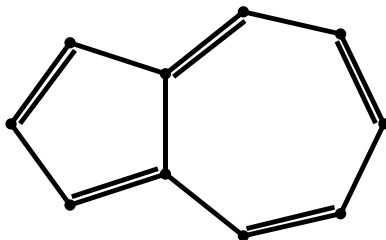
D-graph	Orbits	r_{ij}	Solutions
	$(m_{01}, m'_{01}, m_{12}, m'_{12})$	$(2, 2, 2, 2)$	$(4, 4, 4, 4)$

Azulene

naphthalene



azulene





Question

“We don’t yet know whether and how the electron mobility might manifest itself among azulenes embedded within a fullerene-style network.

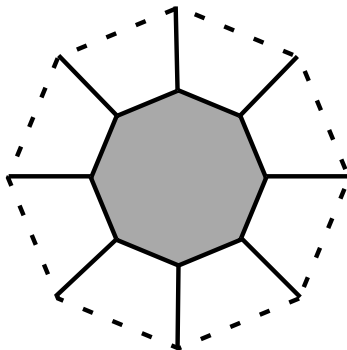
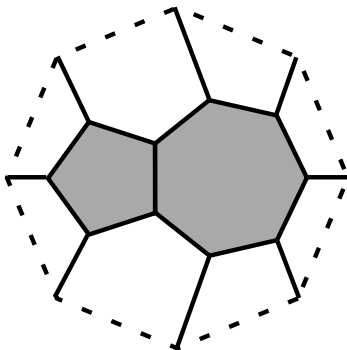
How many variations of such networks are theoretically possible?”

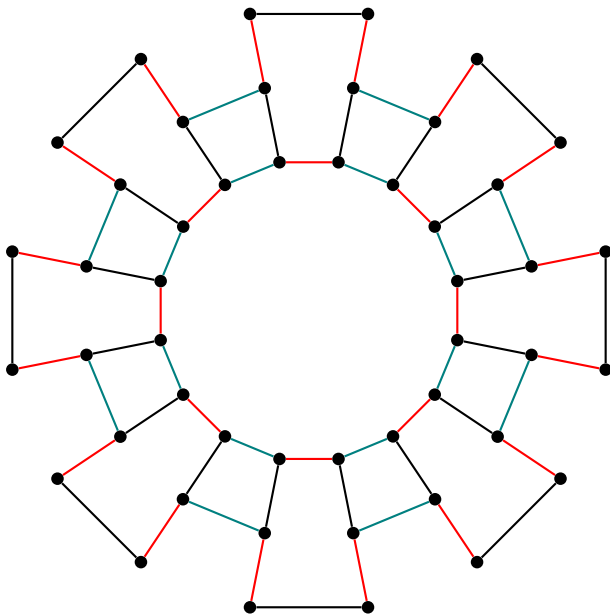
Edward Kirby

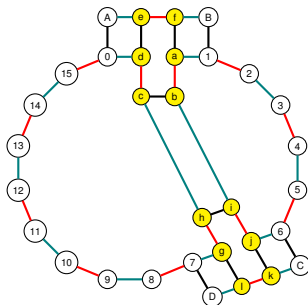
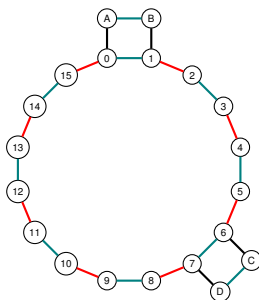


Translation

A cubic tiling containing a single orbit of pairwise disjoint azulenoids that together cover all vertices.



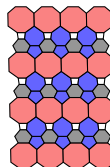
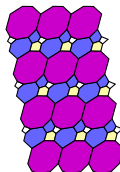
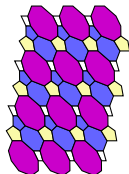
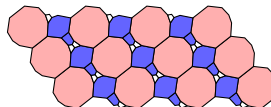
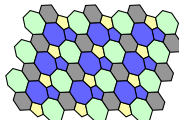
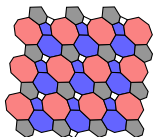




Result

- There are 12 possible multisets of face sizes appearing around the octagon.
- There are 272 assignments of face sizes around the octagon.
- There are 255 octagon tilings satisfying the requirements for azulenoïd tilings.
- There are 1,274 azulenoïd tilings.

Translational-symmetry azulenoid tilings



Why not generate all Delaney-Dress symbols?

n	Delaney-Dress graphs	time	time	rate
1	1	0.0s	0.0s	
2	7	0.0s	0.0s	
3	3	0.0s	0.0s	
4	22	0.0s	0.0s	
5	13	0.0s	0.0s	
6	70	0.0s	0.0s	
7	67	0.0s	0.0s	
8	315	0.0s	0.0s	
9	393	0.0s	0.0s	
10	1,577	0.0s	0.0s	
11	2,515	0.0s	0.0s	
12	9,480	0.1s	0.1s	94,800.00/s
13	17,205	0.1s	0.1s	172,050.00/s
14	61,594	0.3s	0.3s	205,313.33/s
15	123,953	0.4s	0.4s	309,882.50/s
16	433,030	1.6s	1.6s	270,643.75/s
17	931,729	2.5s	2.5s	372,691.60/s
18	3,196,841	9.1s	9.1s	351,301.21/s
19	7,258,011	16.3s	16.3s	445,276.75/s
20	24,630,262	55.0s	55.0s	447,822.95/s
21	58,309,071	105.9s	1.8m	550,605.01/s
22	196,266,434	345.5s	5.8m	568,064.93/s
23	481,330,615	722.2s	12.0m	666,478.28/s
24	1,610,942,856	2,329.2s	38.8m	691,629.25/s
25	4,071,117,829	5,184.9s	86.4m	785,187.34/s
26	13,569,014,653	16,422.1s	4.6h	826,265.50/s
27	35,202,390,477	38,273.5s	10.6h	919,758.85/s
28	116,994,675,348	121,796.3s	1.4d	960,576.60/s
29	310,624,700,725	295,889.0s	3.4d	1,049,801.45/s
30	1,030,455,432,427	949,823.0s	11.0d	1,084,892.06/s

Why not generate all Delaney-Dress symbols?

n	Delaney-Dress graphs	time	time	rate
1	1	0.0s	0.0s	
2	7	0.0s	0.0s	
3	3	0.0s	0.0s	
4	22	0.0s	0.0s	
5	13	0.0s	0.0s	
6	70	0.0s	0.0s	
7	67	0.0s	0.0s	
8	315	0.0s	0.0s	
9	393	0.0s	0.0s	
10	1,577	0.0s	0.0s	
11	2,515	0.0s	0.0s	
12	9,480	0.1s	0.1s	94,800.00/s
13	17,205	0.1s	0.1s	172,050.00/s
14	61,594	0.3s	0.3s	205,313.33/s
15	123,953	0.4s	0.4s	309,882.50/s
16	433,030	1.6s	1.6s	270,643.75/s
17	931,729	2.5s	2.5s	372,691.60/s
18	3,196,841	9.1s	9.1s	351,301.21/s
19	7,258,011	16.3s	16.3s	445,276.75/s
20	24,630,262	55.0s	55.0s	447,822.95/s
21	58,309,071	105.9s	1.8m	550,605.01/s
22	196,266,434	345.5s	5.8m	568,064.93/s
23	481,330,615	722.2s	12.0m	666,478.28/s
24	1,610,942,856	2,329.2s	38.8m	691,629.25/s
25	4,071,117,829	5,184.9s	86.4m	785,187.34/s
26	13,569,014,653	16,422.1s	4.6h	826,265.50/s
27	35,202,390,477	38,273.5s	10.6h	919,758.85/s
28	116,994,675,348	121,796.3s	1.4d	960,576.60/s
29	310,624,700,725	295,889.0s	3.4d	1,049,801.45/s
30	1,030,455,432,427	949,823.0s	11.0d	1,084,892.06/s

Why not generate all Delaney-Dress symbols?

n	Delaney-Dress graphs	time	rate
17	931,729	2.5s	372,691.60/s

n	Delaney-Dress symbols	time	time
17	13,105	304,083.2s	3.5d



Outline

- 1 Introduction
 - Main characters
- 2 Delaney–Dress symbols
 - Construction
 - Definition
 - Working with Delaney–Dress symbols
- 3 Applications
 - Tile- k -transitive tilings
 - Isotoxal tilings
 - Azulenoids
- 4 Visualisation
 - Technique
 - Examples
- 5 Local symmetry-preserving operations
 - Introduction
 - Operations and Delaney–Dress symbols
 - Compression



Software

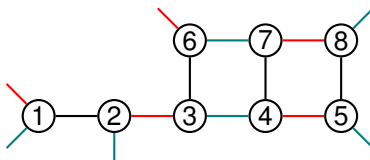
- Gavrog (<http://gavrog.org/>)
 - Euclidean
 - 3D and 2D
- Tegula (<https://ab.inf.uni-tuebingen.de/software/tegula>)
 - Euclidean, spherical, and hyperbolic
 - 2D only



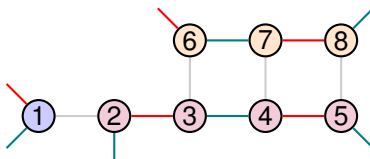
Visualisation steps for the Euclidean case

- 1 Identify the orbits and their branching numbers
- 2 Glue chambers into one fundamental polygon
→ never close a branched cycle
- 3 Shape the fundamental polygon
→ branching and mirrors give the angles
- 4 Identify the generators of the symmetry group
- 5 Grow the tiling

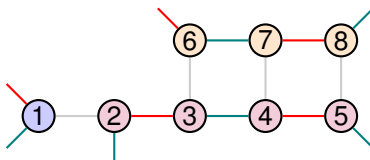




	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

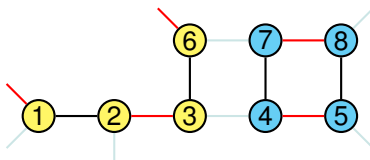


	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

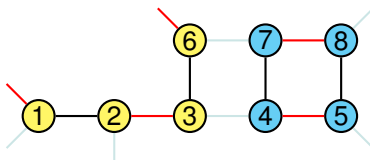


v_{01} 3 1 1

	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

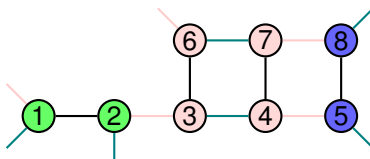


	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

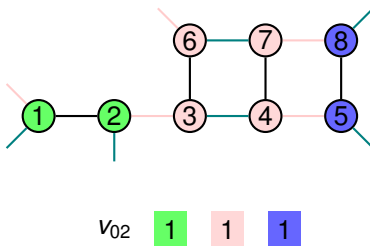


V_{12} 1 3

	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

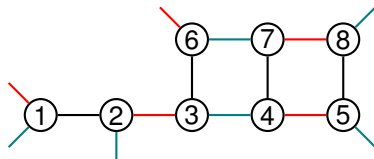


	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

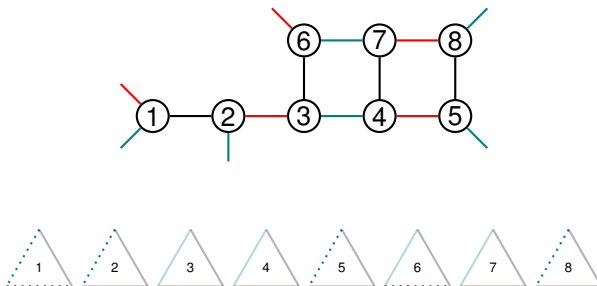


	m_{01}	m_{12}
1	3	4
2	4	4
3	4	4
4	4	6
5	4	6
6	3	4
7	3	6
8	3	6

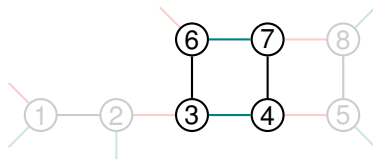
Gluing the chambers together



Gluings the chambers together



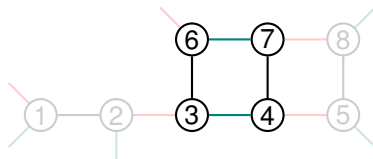
Gluing the chambers together



$$v_{02} = 1$$



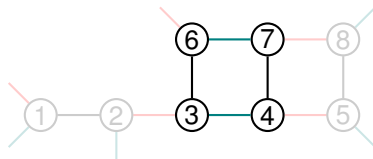
Gluings the chambers together



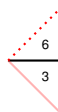
$$v_{02} = 1$$



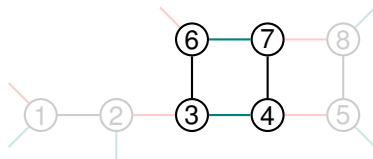
Gluings the chambers together



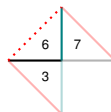
$$v_{02} = 1$$



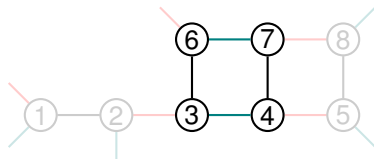
Gluing the chambers together



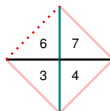
$$v_{02} = 1$$



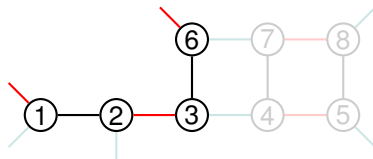
Gluing the chambers together



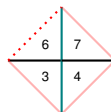
$$v_{02} = 1$$



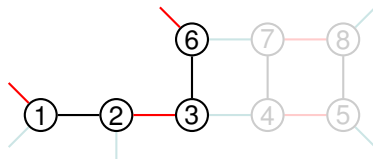
Gluings the chambers together



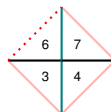
$$v_{12} = 1$$



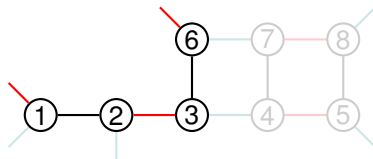
Gluing the chambers together



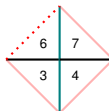
$$v_{12} = 1$$



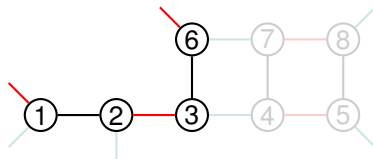
Gluing the chambers together



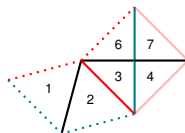
$$v_{12} = 1$$



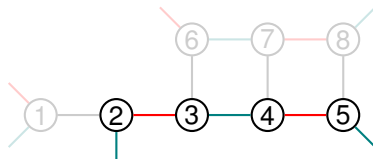
Gluing the chambers together



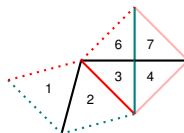
$$v_{12} = 1$$



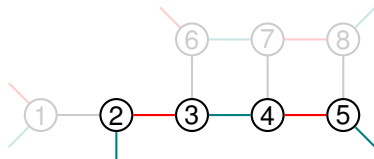
Gluing the chambers together



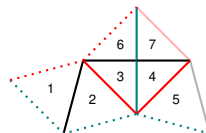
$$v_{01} = 1$$



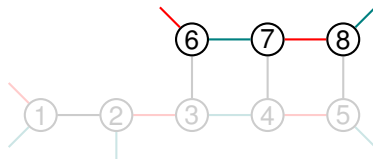
Gluing the chambers together



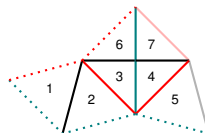
$$v_{01} = 1$$



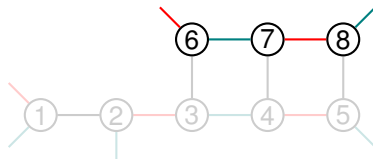
Gluing the chambers together



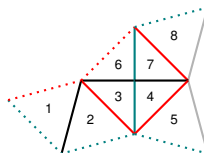
$$v_{01} = 1$$



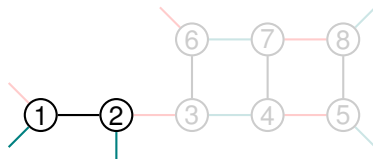
Gluing the chambers together



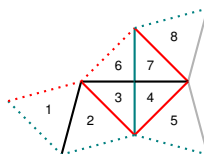
$$v_{01} = 1$$



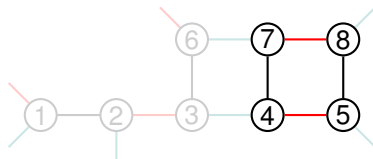
Gluing the chambers together



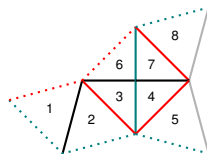
$$v_{02} = 1$$



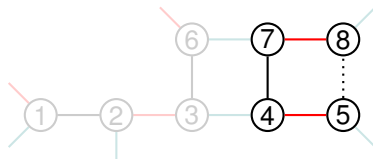
Gluing the chambers together



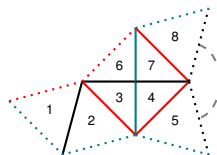
$$v_{12} = 3$$



Gluing the chambers together



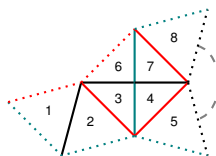
$$v_{12} = 3$$



Gluing the chambers together



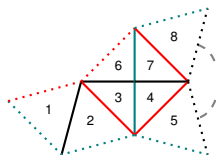
$$v_{01} = 3$$



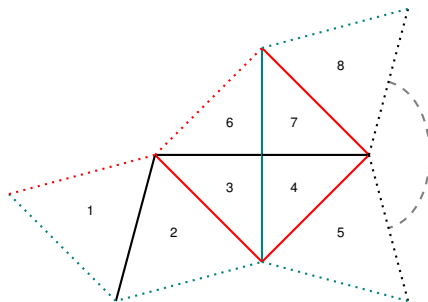
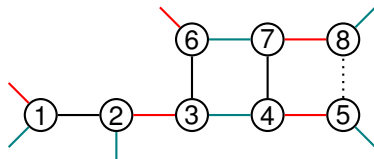
Gluing the chambers together



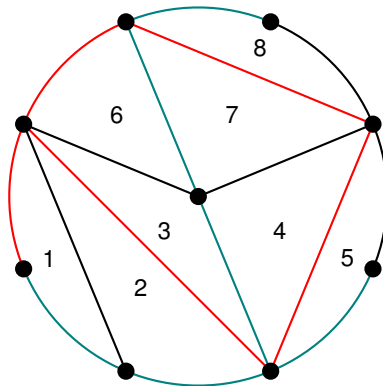
$$v_{02} = 1$$



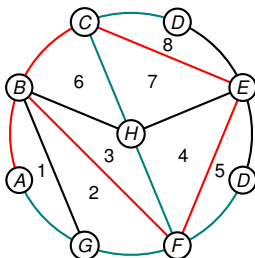
Gluings the chambers together



Tracing the boundary

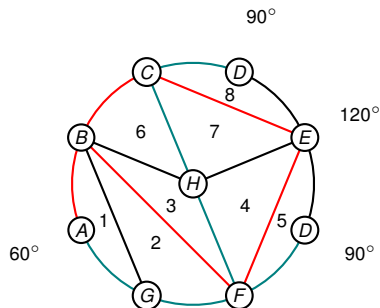


Tracing the boundary



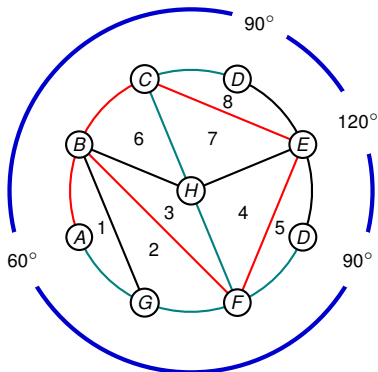
	v_{ij}	f	o	$\frac{2\pi}{v_{ij} \cdot f \cdot o}$
A	3	2	1	60°
B	1	2	1	180°
C	1	2	1	180°
D	1	2	2	90°
E	3	1	1	120°
F	1	2	1	180°
G	1	2	1	180°
H	1	1	1	360°

Tracing the boundary



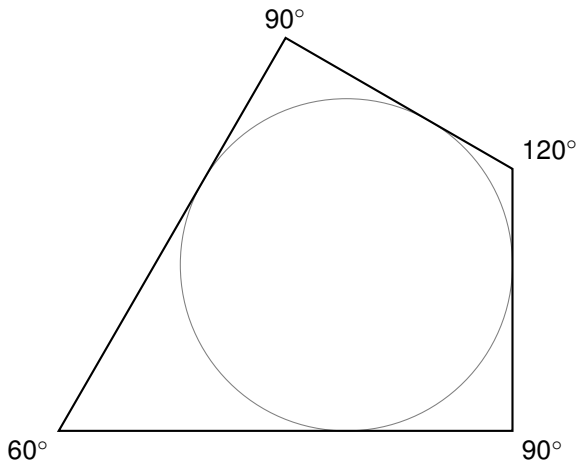
	v_{ij}	f	o	$\frac{2\pi}{v_{ij} \cdot f \cdot o}$
A	3	2	1	60°
B	1	2	1	180°
C	1	2	1	180°
D	1	2	2	90°
E	3	1	1	120°
F	1	2	1	180°
G	1	2	1	180°
H	1	1	1	360°

Tracing the boundary

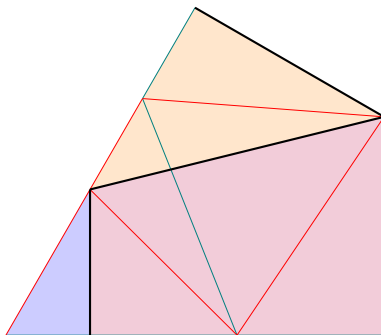


	v_{ij}	f	o	$\frac{2\pi}{v_{ij} \cdot f \cdot o}$
A	3	2	1	60°
B	1	2	1	180°
C	1	2	1	180°
D	1	2	2	90°
E	3	1	1	120°
F	1	2	1	180°
G	1	2	1	180°
H	1	1	1	360°

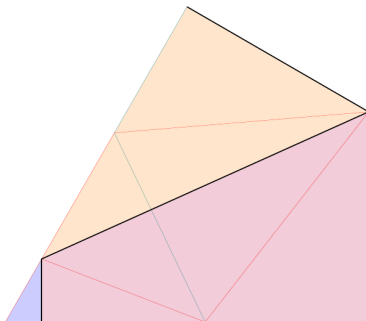
Shaping the boundary



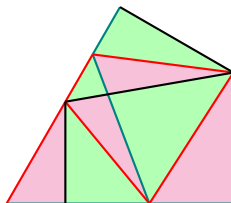
Shaping the boundary



Shaping the boundary

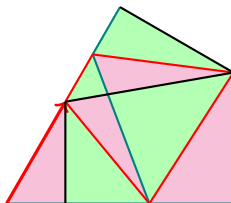


Find symmetry generators



- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



■ Reflection across a line

■ continuation of the reflection

■ continuation of the reflection

■ Rotation by 120° about point c

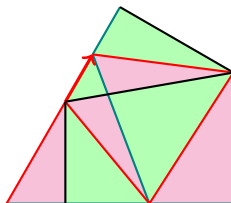
■ Inverse of rotation

■ Reflection across a line

■ continuation of the reflection

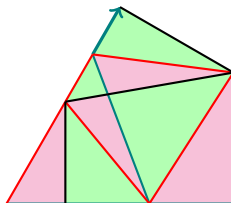
■ continuation of the reflection

Find symmetry generators



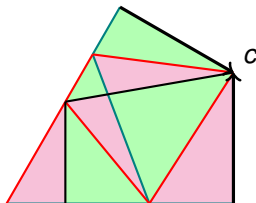
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



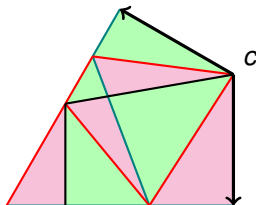
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



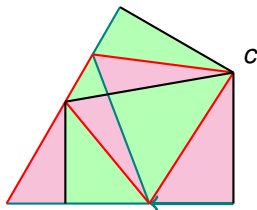
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



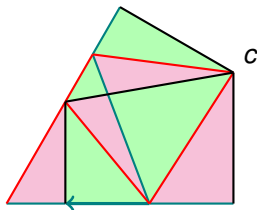
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- Reflection across a line
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



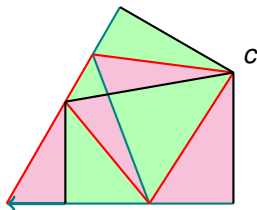
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators



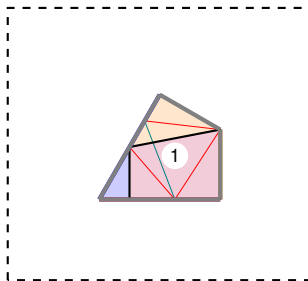
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

Find symmetry generators

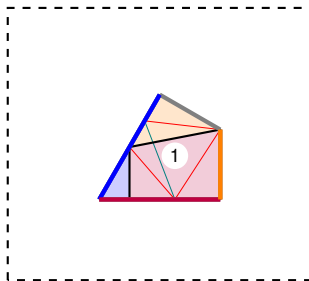


- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection
- **Rotation by 120° about point c**
- Inverse of rotation
- **Reflection across a line**
- continuation of the reflection
- continuation of the reflection

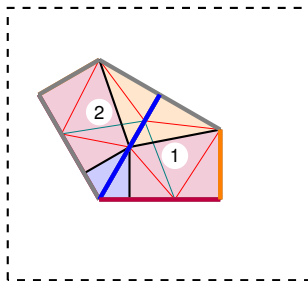
Growing the tiling



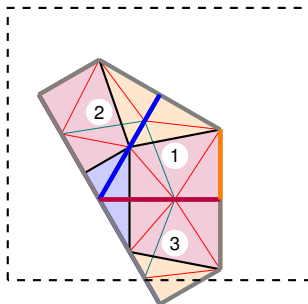
Growing the tiling



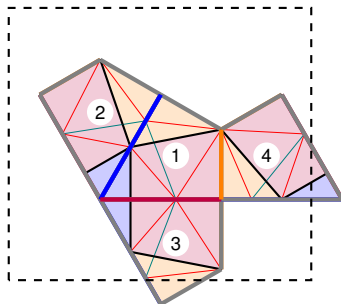
Growing the tiling



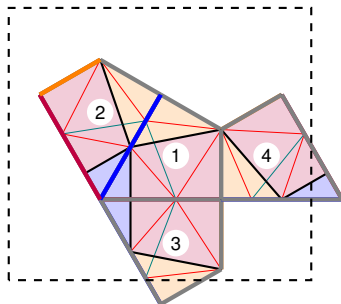
Growing the tiling



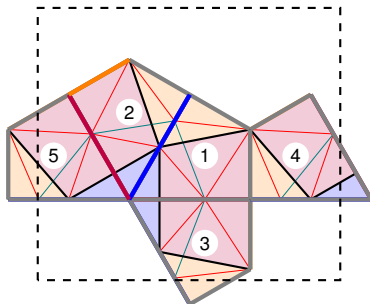
Growing the tiling



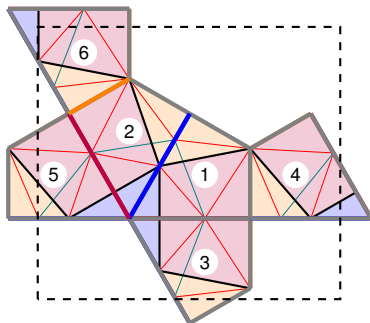
Growing the tiling



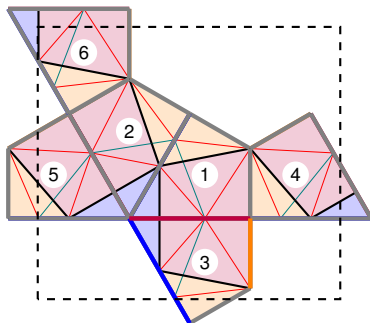
Growing the tiling



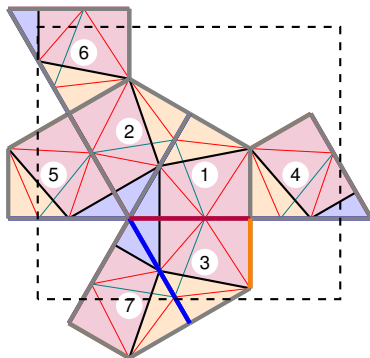
Growing the tiling



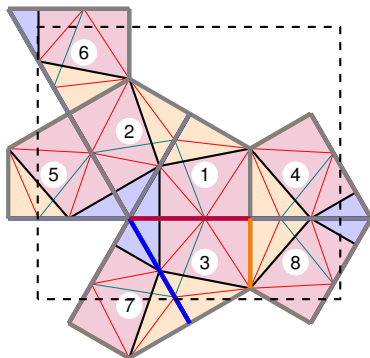
Growing the tiling



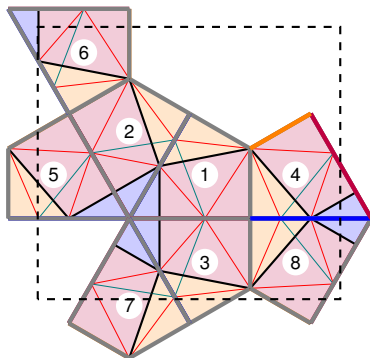
Growing the tiling



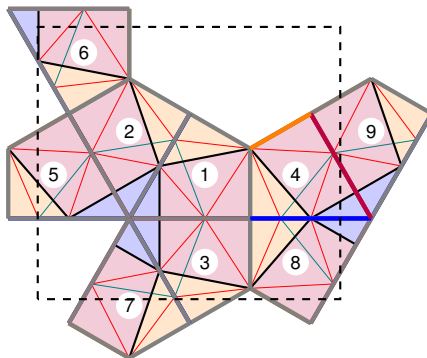
Growing the tiling



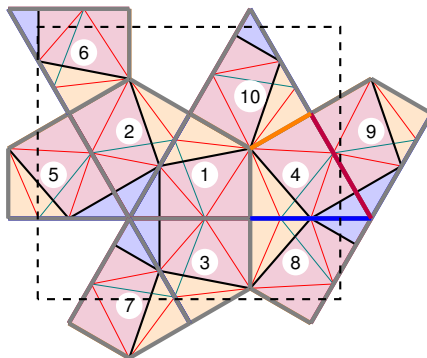
Growing the tiling



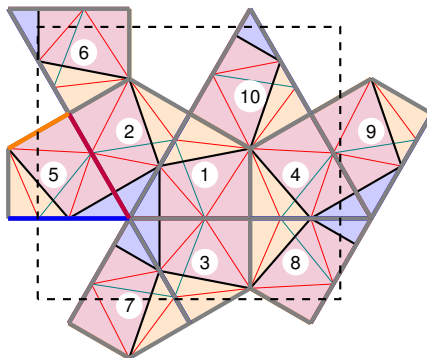
Growing the tiling



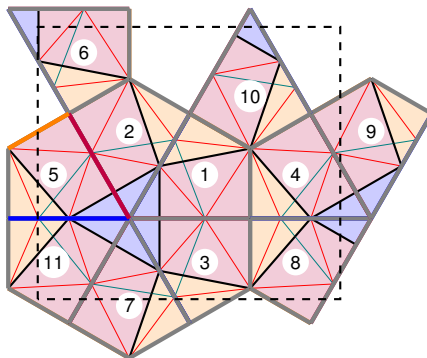
Growing the tiling



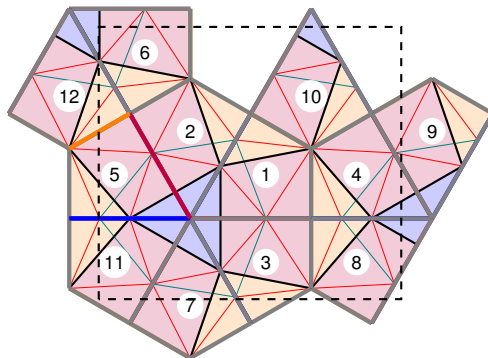
Growing the tiling



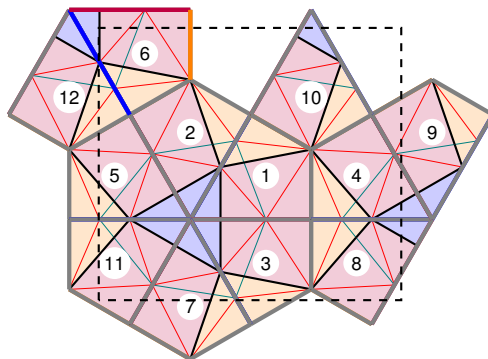
Growing the tiling



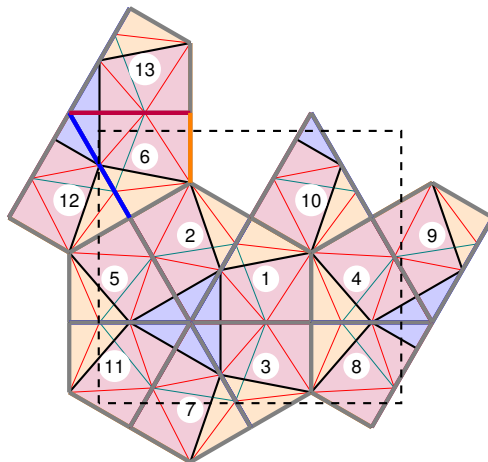
Growing the tiling



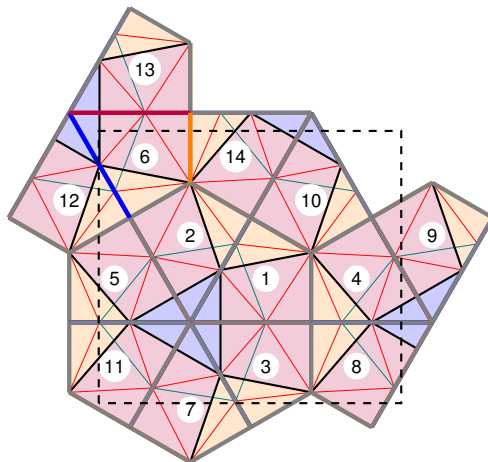
Growing the tiling



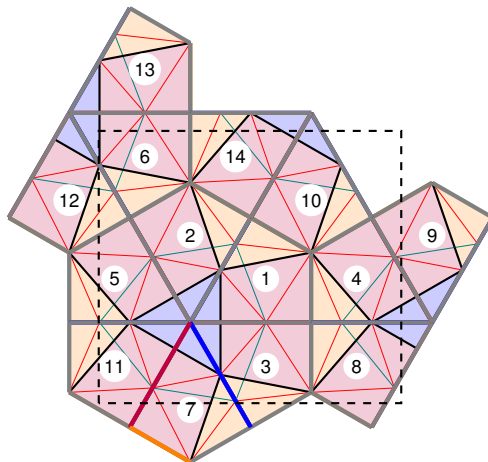
Growing the tiling



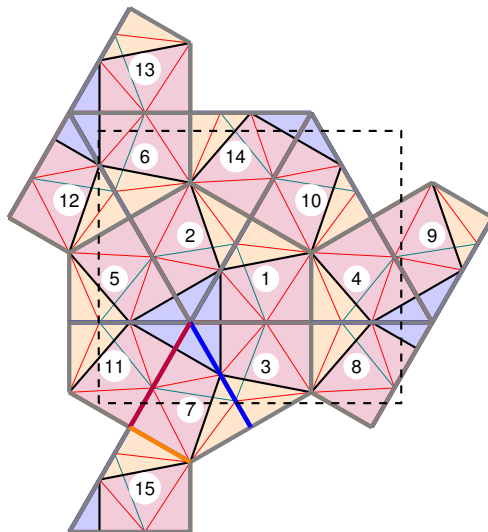
Growing the tiling



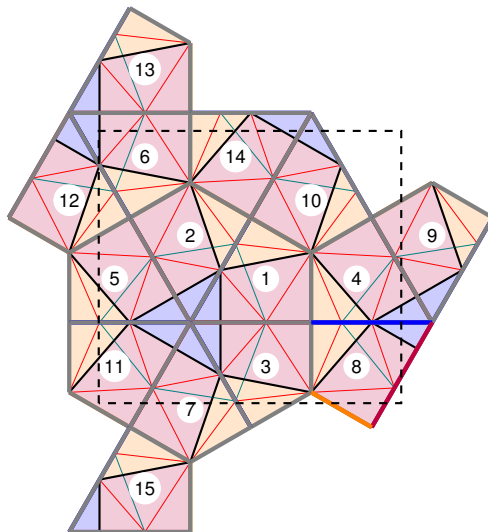
Growing the tiling



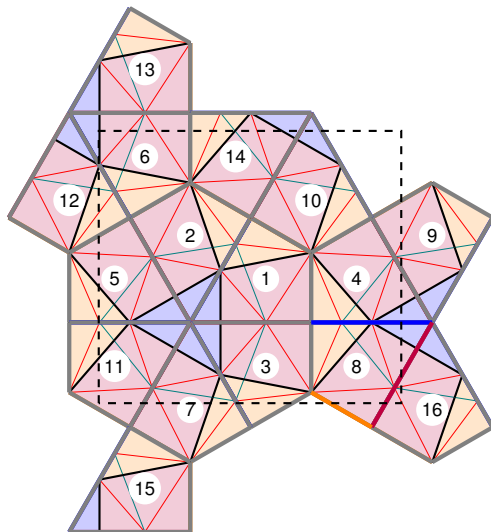
Growing the tiling



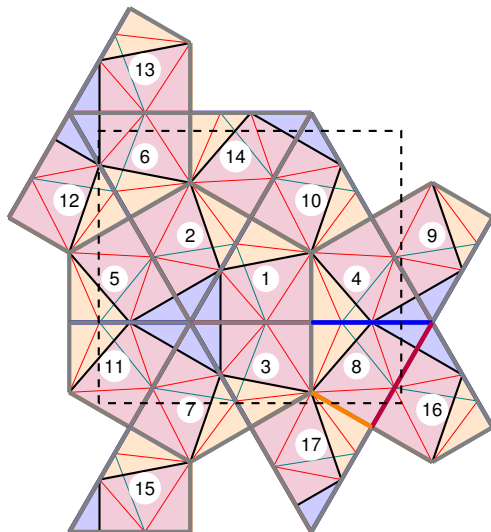
Growing the tiling



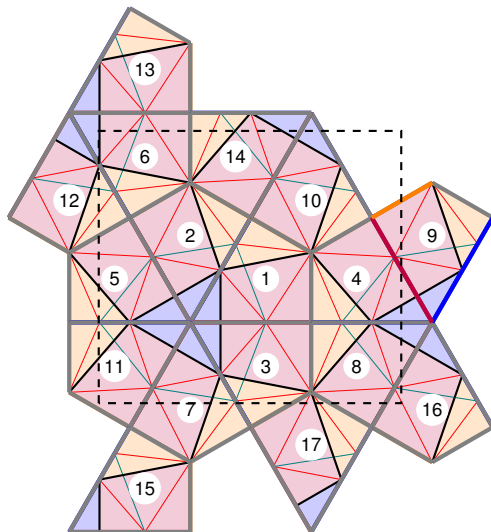
Growing the tiling



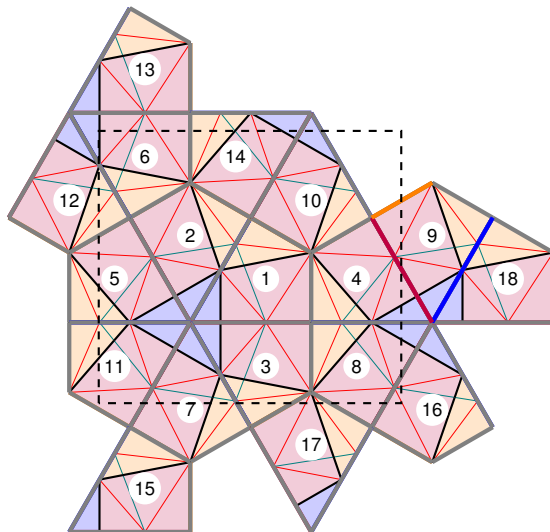
Growing the tiling



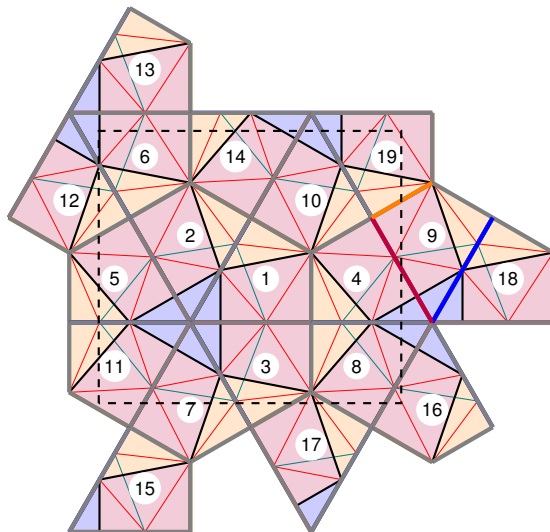
Growing the tiling



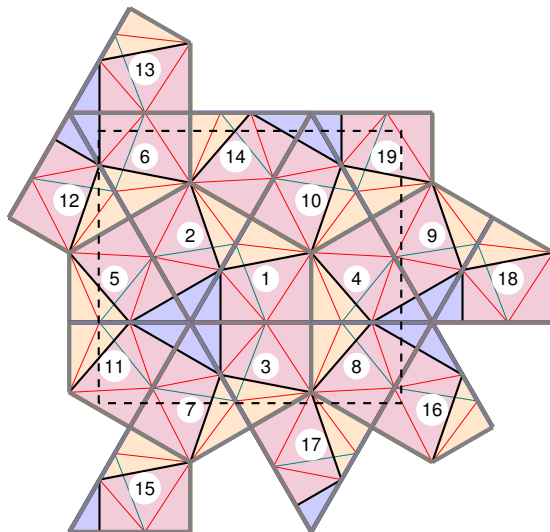
Growing the tiling

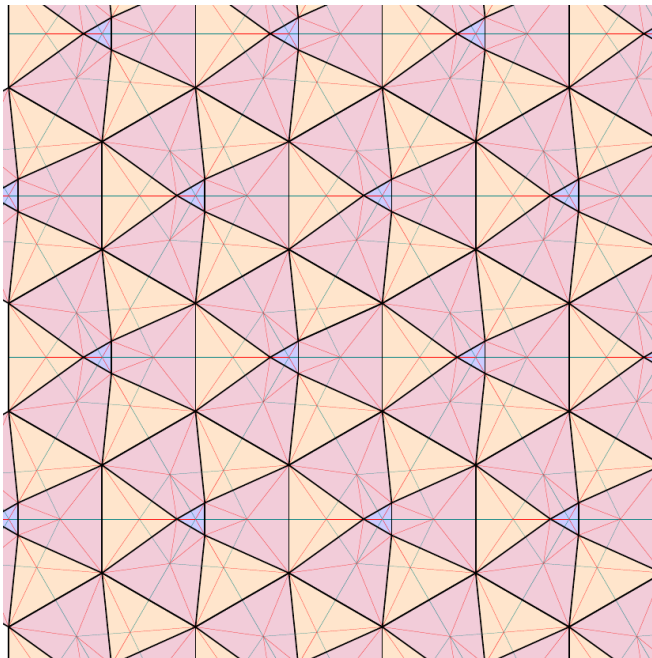


Growing the tiling



Growing the tiling



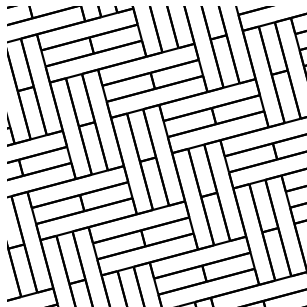
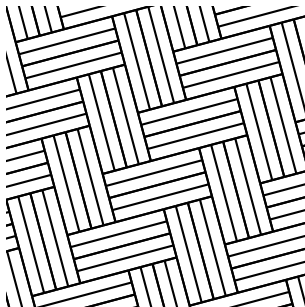


Collatz's question

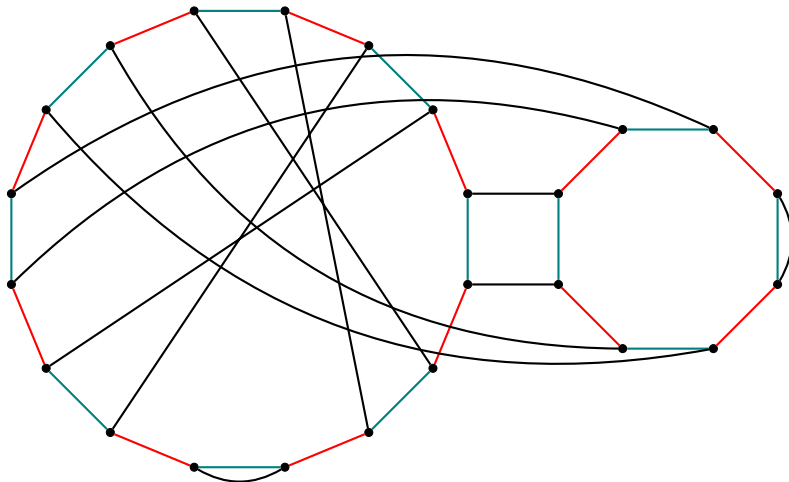
Question posed to the Bielefeld group by Lothar Collatz:

Question

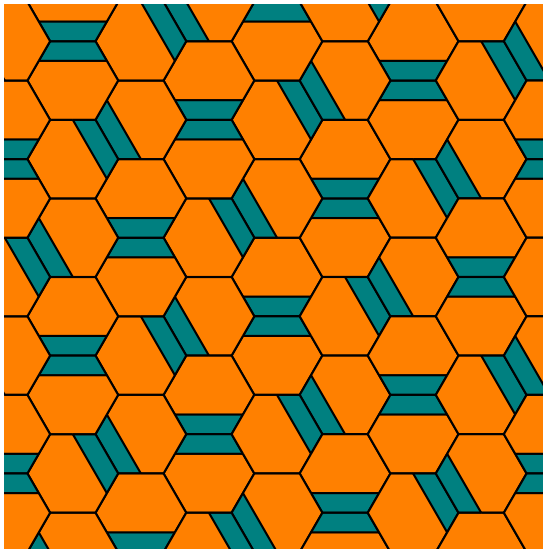
Are these two tilings equivariantly equivalent?



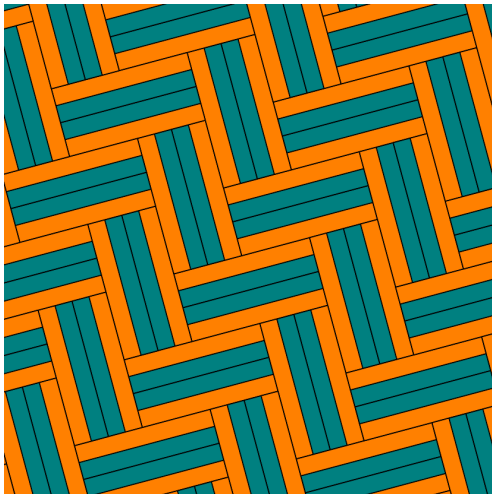
Collatz's question



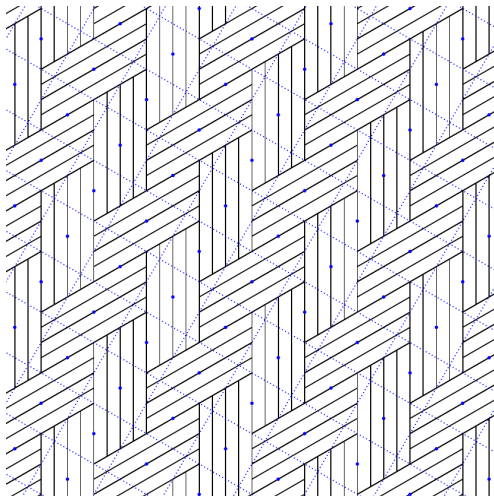
Collatz's question



Collatz's question



Collatz's question

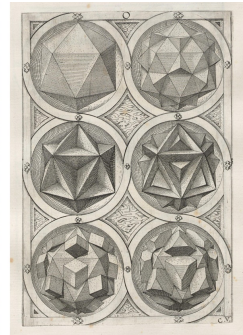
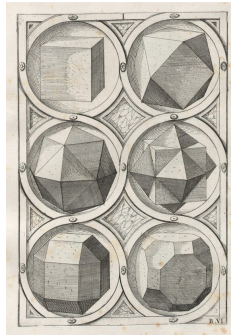
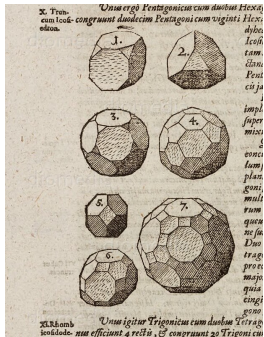


Outline

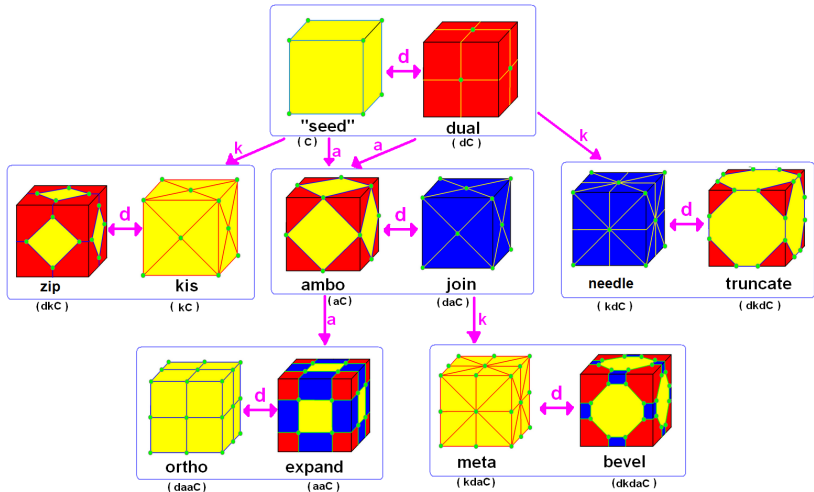
- 1 Introduction
 - Main characters
- 2 Delaney–Dress symbols
 - Construction
 - Definition
 - Working with Delaney–Dress symbols
- 3 Applications
 - Tile- k -transitive tilings
 - Isotoxal tilings
 - Azulenoids
- 4 Visualisation
 - Technique
 - Examples
- 5 Local symmetry-preserving operations
 - Introduction
 - Operations and Delaney–Dress symbols
 - Compression



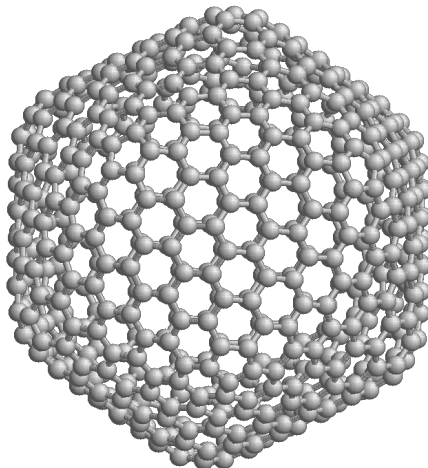
Operations



Operations



Operations



Operations

- As with tilings: no way to know if enumerations are complete
- Need for a systematic approach
- Criteria:
 - Local: small definition, but repeated throughout the whole structure
 - Symmetry-preserving: the result has at least the same symmetry



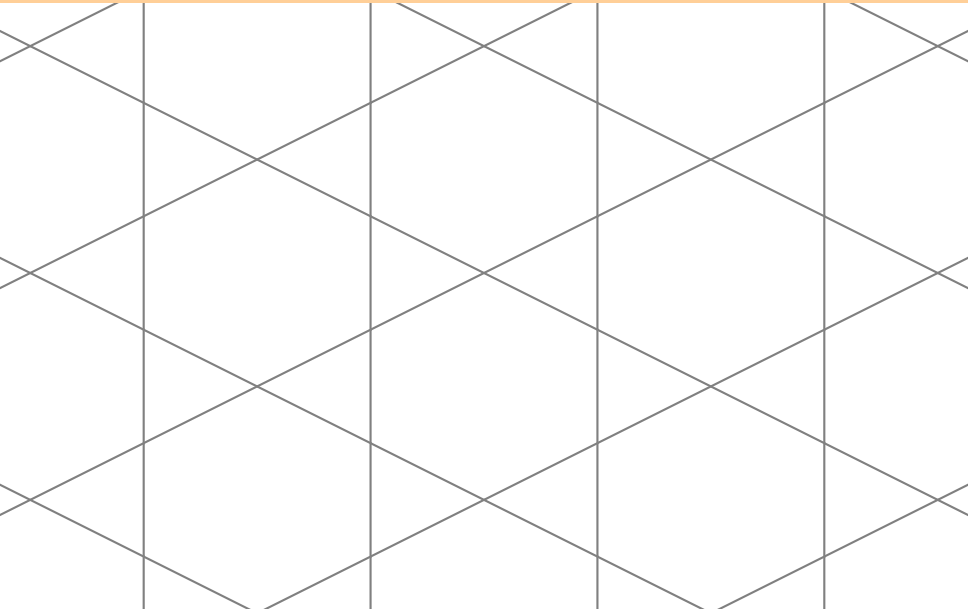
Local symmetry-preserving operations

Take your favourite tiling with $p6m$ symmetry

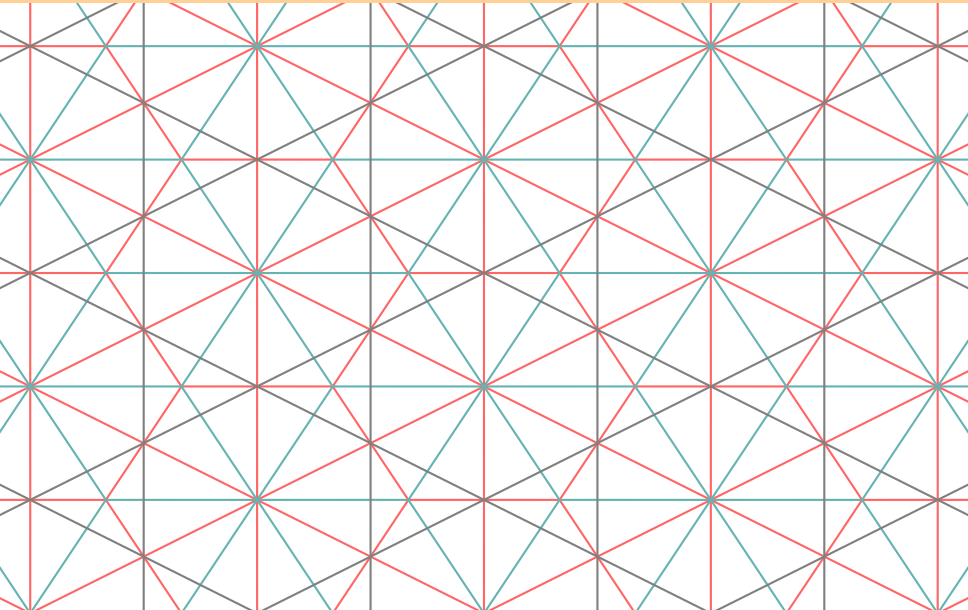
Local symmetry-preserving operations



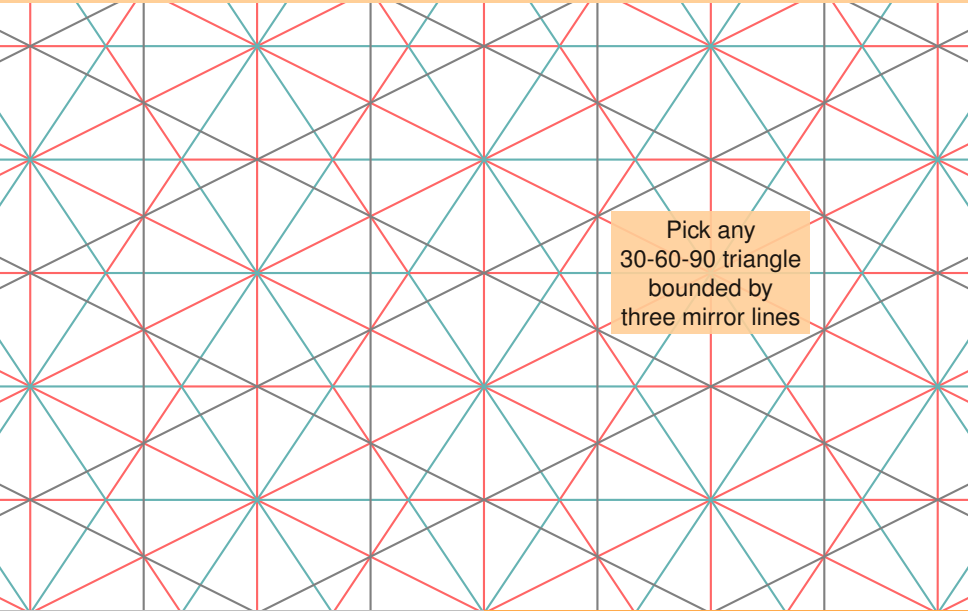
Local symmetry-preserving operations



Local symmetry-preserving operations

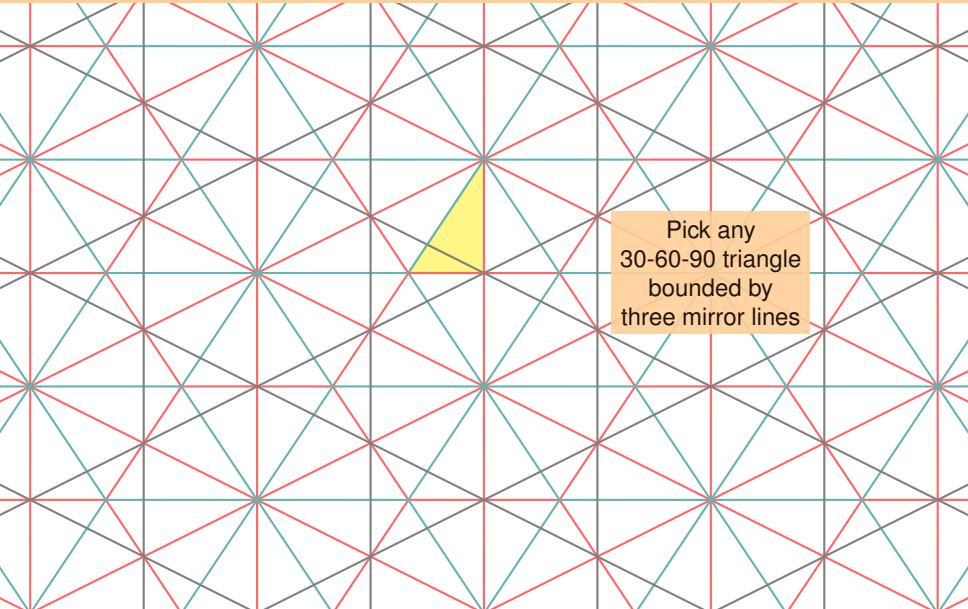


Local symmetry-preserving operations

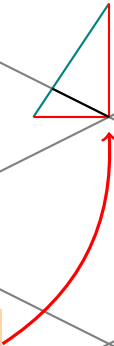


Pick any
30-60-90 triangle
bounded by
three mirror lines

Local symmetry-preserving operations

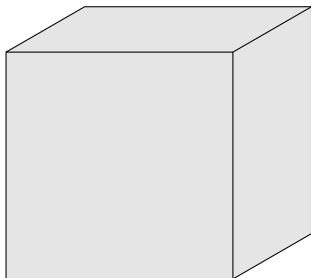


Local symmetry-preserving operations

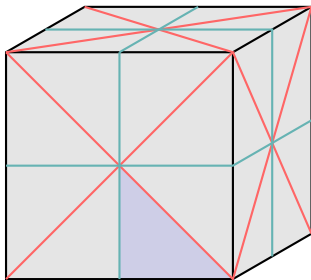


This set of chambers
is the operation

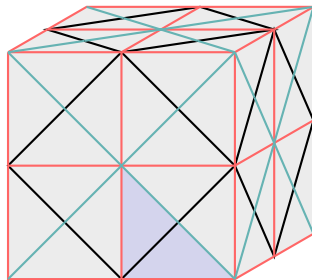
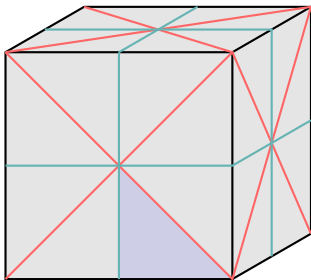
Local symmetry-preserving operations



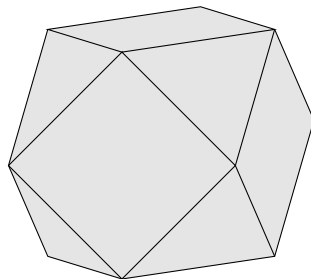
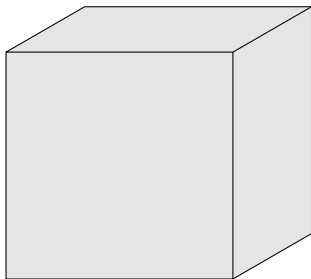
Local symmetry-preserving operations



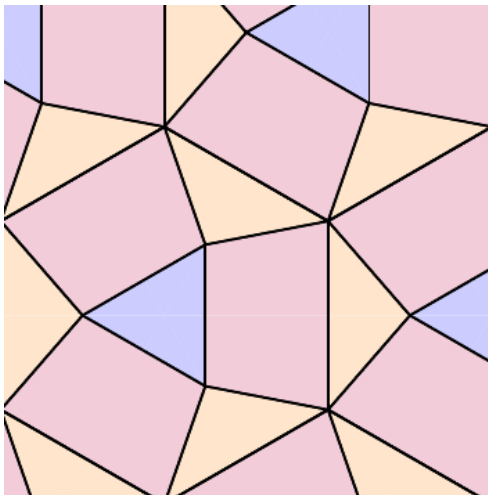
Local symmetry-preserving operations



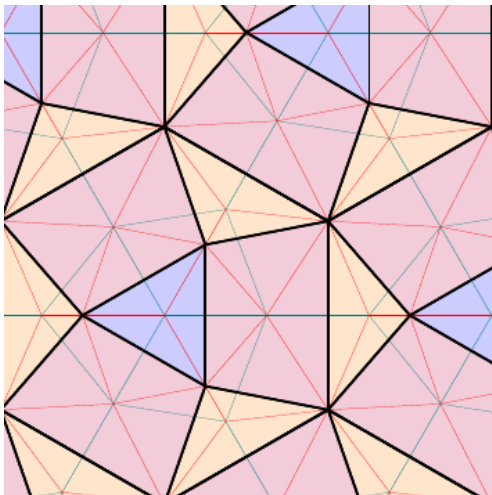
Local symmetry-preserving operations



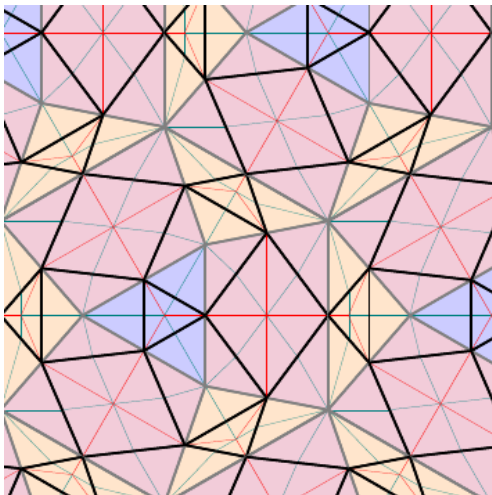
Local symmetry-preserving operations



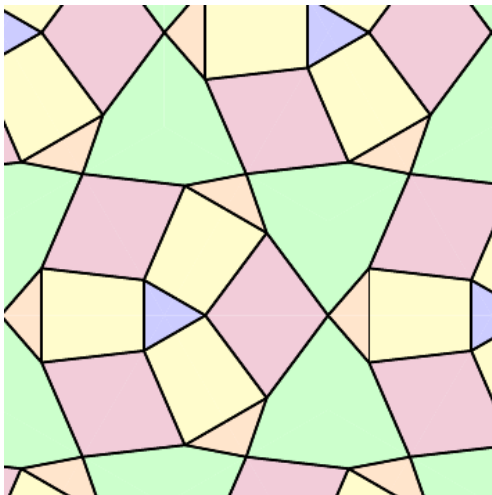
Local symmetry-preserving operations



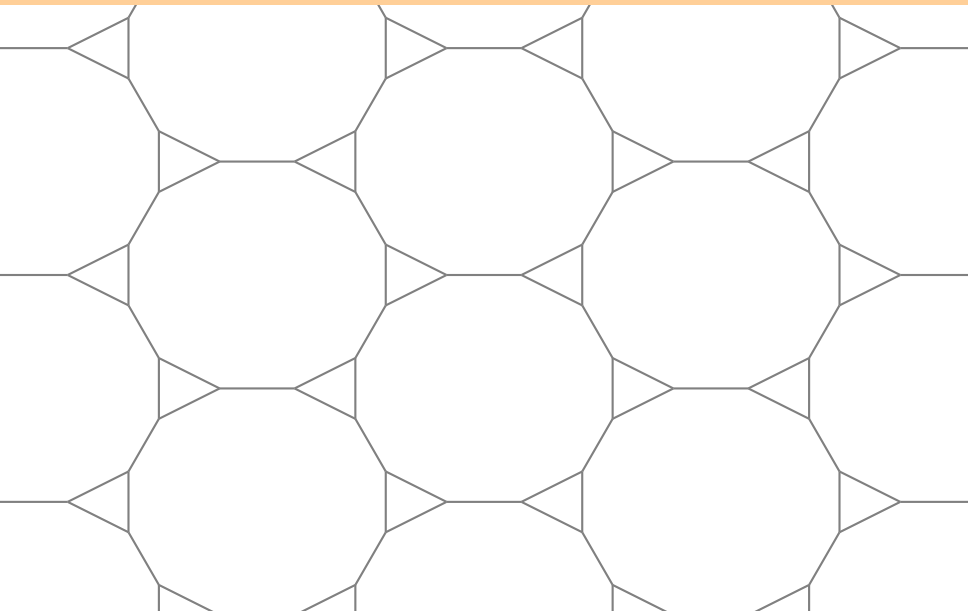
Local symmetry-preserving operations



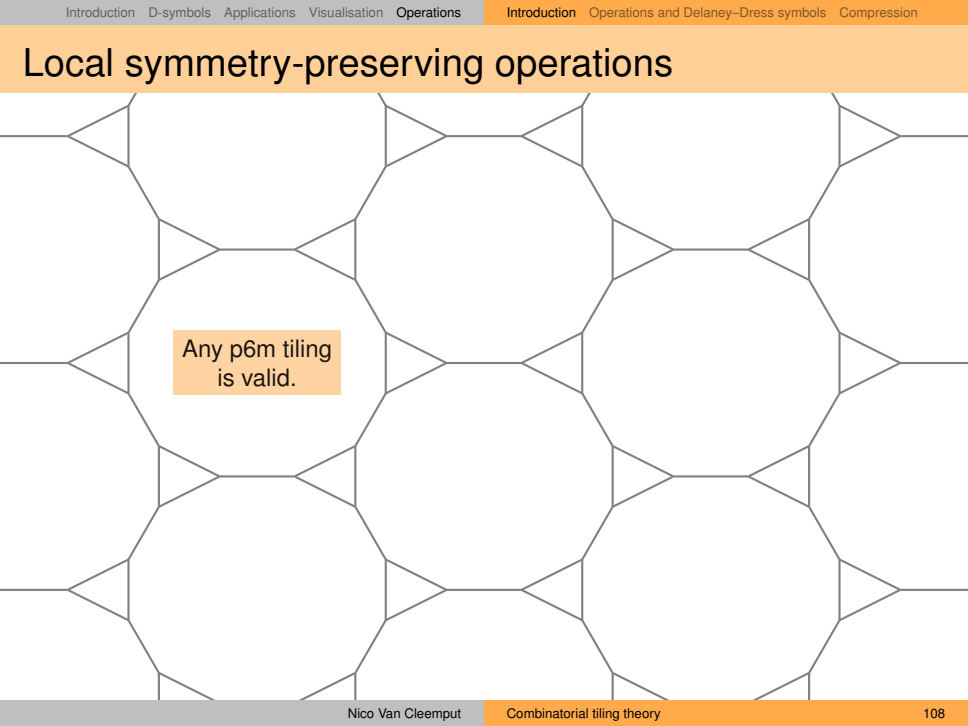
Local symmetry-preserving operations



Local symmetry-preserving operations



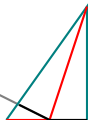
Local symmetry-preserving operations



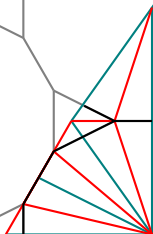
Any p6m tiling
is valid.

Local symmetry-preserving operations

Any p6m tiling
is valid.



Local symmetry-preserving operations



Any 30-60-90 triangle bounded by three mirror lines is valid.

Operations and Delaney–Dress symbols

- Let T be a tiling with p6m symmetry group Γ . The operation corresponds to the Delaney–Dress symbol of (T, Γ') where $\Gamma' \cong \text{p6m}$ and $\Gamma' \subseteq \Gamma$.
- The Delaney–Dress graph with marked corner points is sufficient to define the operation.
 - For closed cycles, $m_{ij} = r_{ij}$.
 - For chains that are not corner points, $m_{ij} = r_{ij}$.
 - For corner points, m_{ij} is a multiple of r_{ij} determined by the host.

Some tedious bookkeeping: a $\sigma_i \sigma_j$ -orbit is glued into a $\sigma_{i'} \sigma_{j'}$ -orbit.
- Using the rules above: the operation can also be applied to the Delaney–Dress symbol.



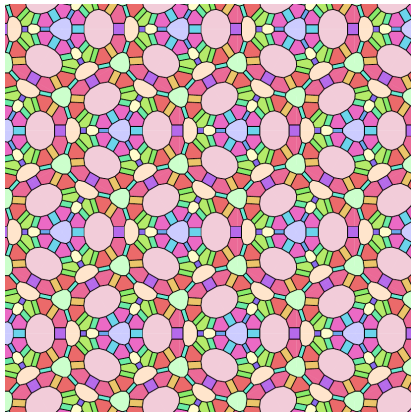
Operations and Delaney–Dress symbols

Restricting to the $p6m$ symmetry group is arbitrary, but not restrictive.

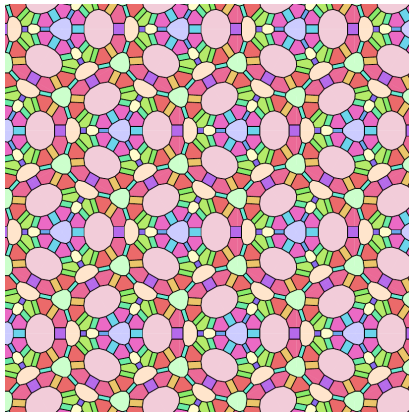
- Assume that O is an operation that cannot be obtained from a tiling with the $p6m$ symmetry group.
- Let (T, Γ) be the regular hexagonal tiling with $p6m$ symmetry group.
- Apply the operation O to (T, Γ) .
- O can be obtained from this new tiling. (Contradiction)



Compression

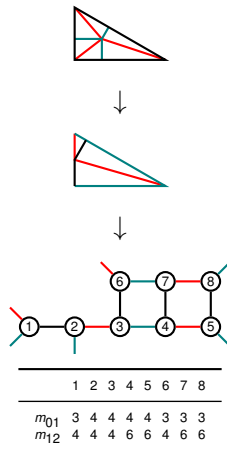
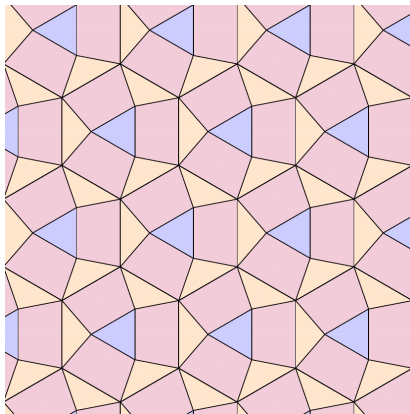


Compression

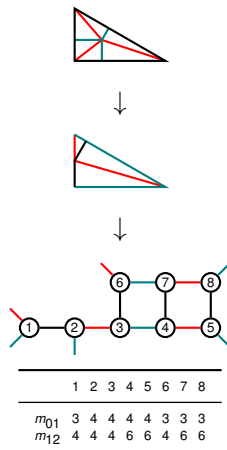
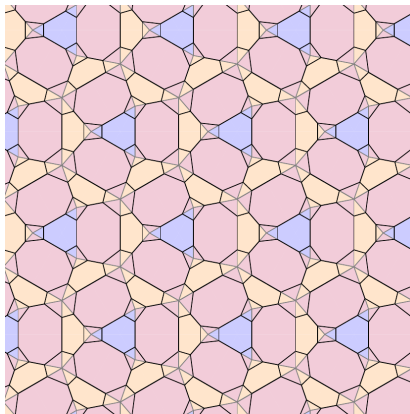


The Delaney–Dress symbol has 144 vertices

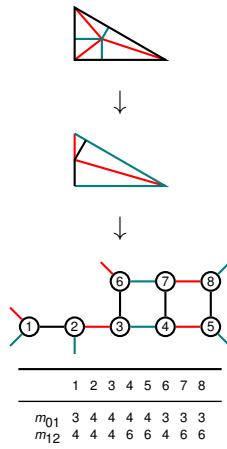
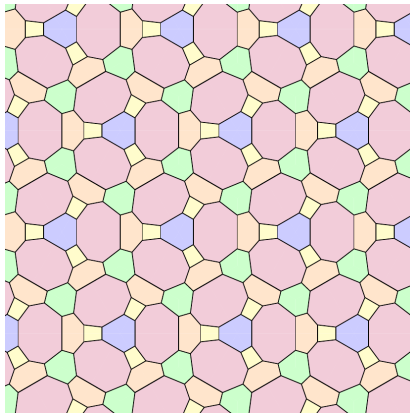
Compression



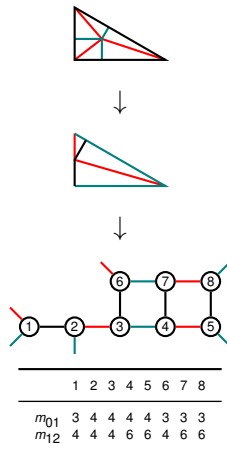
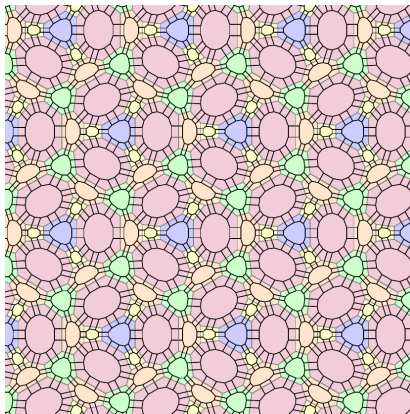
Compression



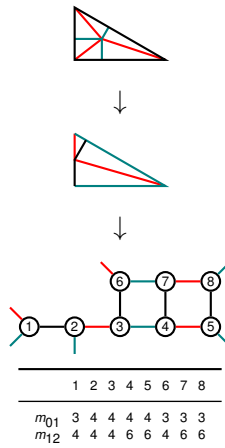
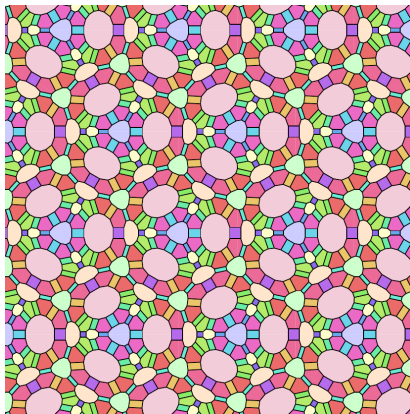
Compression



Compression



Compression



Open questions

- Is ambo (rectification) the only (essential) operation that can increase symmetry? What about 3D operations?
- What are non-equivalent operations that produce isomorphic results for some polyhedra, and non-isomorphic results for others? (e.g. identity vs. dual applied to self-dual polyhedra)
- Is there an efficient algorithm to decompose a given polyhedron/tiling into a base and a sequence of decorations?

