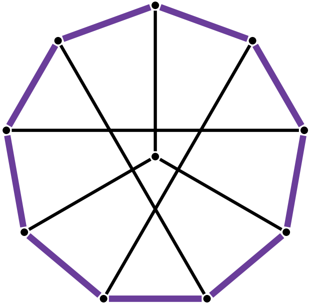
Shortness coefficient of cyclically 4-edge-connected cubic graphs

On-Hei S. Lo Jens M. Schmidt
Nico Van Cleemput Carol T. Zamfirescu

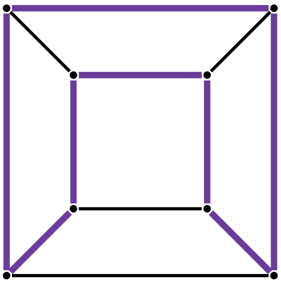
Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics, Computer Science and Statistics
Ghent University

Circumference



The circumference $circ(G)$ is the length of a longest cycle.

Hamiltonicity

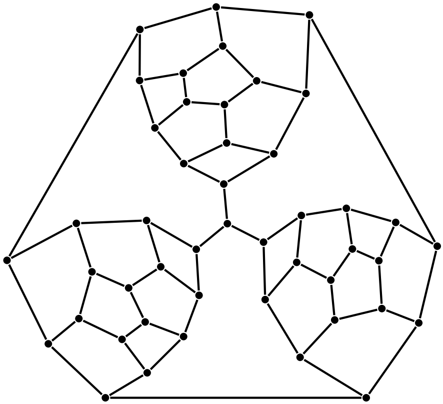


A graph G is hamiltonian if $circ(G) = |V(G)|$.

Hamiltonicity of classes of graphs

- Tait conjectured in 1884 that every cubic polyhedron is hamiltonian.
- The conjecture became famous because it implied the Four Colour Theorem (at that time still the Four Colour Problem)

Hamiltonicity of classes of graphs



The first to construct a counterexample was Tutte in 1946

Hamiltonicity of classes of graphs

Theorem (Tutte, 1956)

Every 4-connected polyhedron is hamiltonian.

Hamiltonicity of classes of graphs

How far is a class of graphs from being hamiltonian?

Shortness coefficient

The **shortness coefficient** of $\mathcal{G}$ is defined as

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\mathrm{circ}(G)}{|V(G)|}$$

with lim inf taken over all sequences of graphs G_n in $\mathcal{G}$ such that $|V(G_n)| \to \infty$ for $n \to \infty$.

Shortness coefficient

$$\rho(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \frac{\mathrm{circ}(G)}{|V(G)|}$$

- $0 \leq \rho(\mathcal{G}) \leq 1$
- every graph in $\mathcal{G}$ is hamiltonian $\Rightarrow \rho(\mathcal{G}) = 1$

Known results

Theorem (Moon and Moser, 1963)

The shortness coefficient of the class of 3-connected planar graphs is 0.

Theorem (Tutte, 1956)

The shortness coefficient of the class of 4-connected planar graphs is 1.

Known results

Theorem (Bondy and Simonovits, 1980)

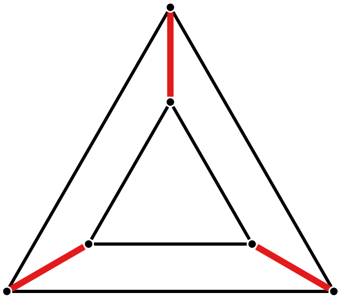
The shortness coefficient of the class of 3-connected cubic graphs is 0.

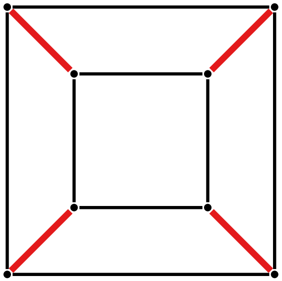
Theorem (Walther, 1969)

The shortness coefficient of the class of 3-connected cubic planar graphs is 0.

Cyclically k-edge-connected

A graph G is cyclically k-edge-connected if for every edge-cut S of G with less than k edges at most one component of $G - S$ contains a cycle.



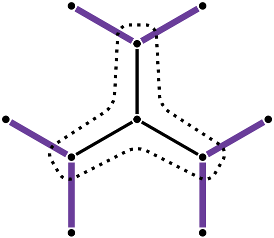


Cyclically k-edge-connected

- For $k \in \{1, 2, 3\}$ being cyclically k-edge-connected and being k-connected are equivalent for cubic graphs.
- $\mathcal{C}k$ is the class of cyclically k-edge-connected cubic graphs.
- $\mathcal{C}k\mathcal{P}$ is the class of cyclically k-edge-connected planar cubic graphs.

Outline

Known bounds



$$circ(G) \geq \frac{3}{4}|V(G)|$$

Known bounds

Theorem (Grünbaum and Malkevitch, 1976)

$\rho(\mathcal{C}4\mathcal{P}) \leq \frac{76}{77}$

Theorem (Lo and Schmidt, 2018)

$\rho(\mathcal{C}4\mathcal{P}) \leq \frac{52}{53}$

Question

$\rho(\mathcal{C}4\mathcal{P}) \leq \frac{41}{42}$?

A new bound

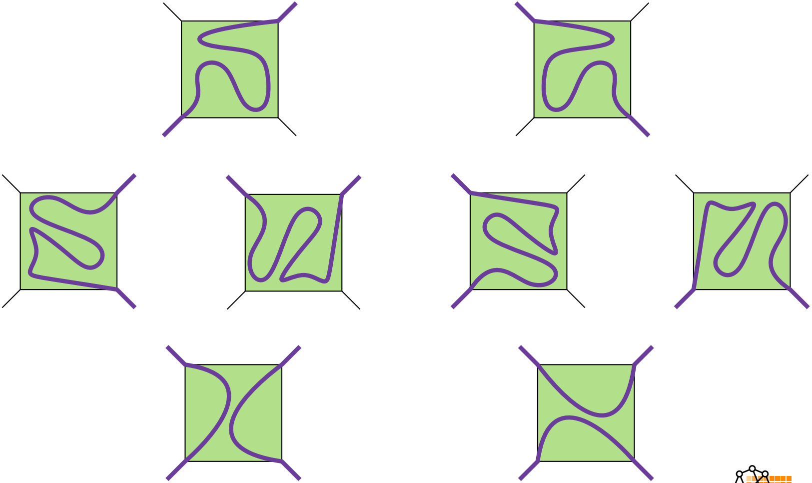
Theorem (Lo, Schmidt, VC, and Zamfirescu)

$\rho(\mathcal{C}4\mathcal{P}) \leq \frac{37}{38}$

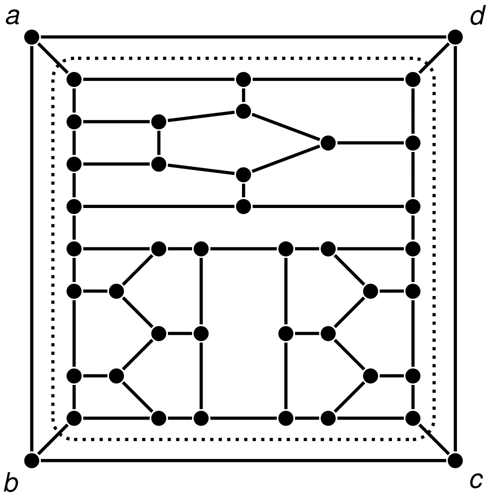
Approach

- Find cyclically 4-edge-connected fragments such that (almost) any intersection with a cycle misses some vertices.
- Combine these fragments to construct an infinite family of graphs obtaining the bound in the limit.

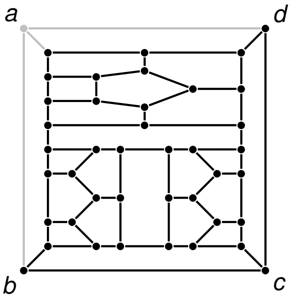
Fragments and cycles

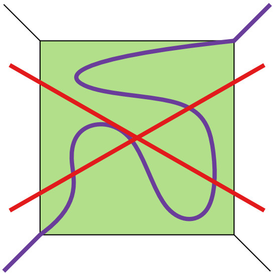


A new bound



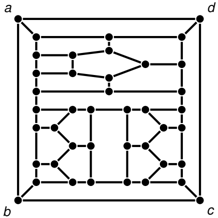
A new bound





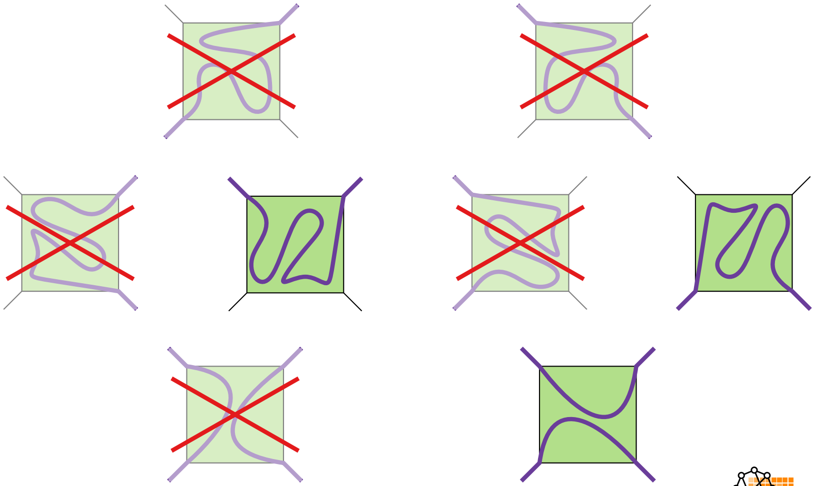
$H - a$ is non-hamiltonian

A new bound

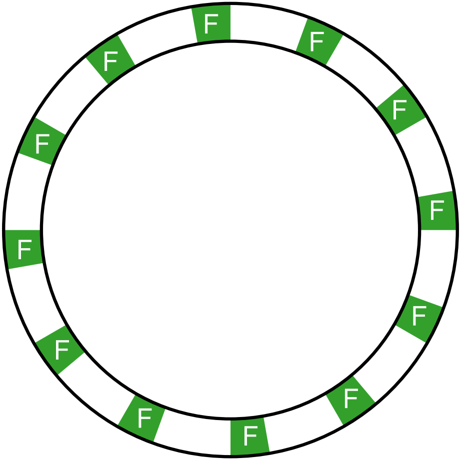


- $H - a$ is non-hamiltonian
- $H - d$ is non-hamiltonian
- $H - a - b$ is non-hamiltonian
- $H - c - d$ is non-hamiltonian
- $H - ab - cd$ is non-hamiltonian

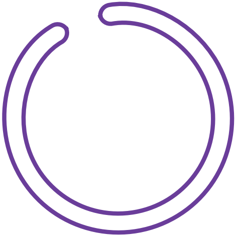
A new bound



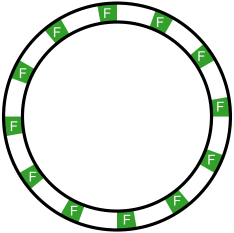
A new bound



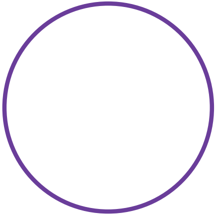
A new bound



misses at least
$k-2$ vertices



k copies of fragment



misses at least
k vertices

A new bound

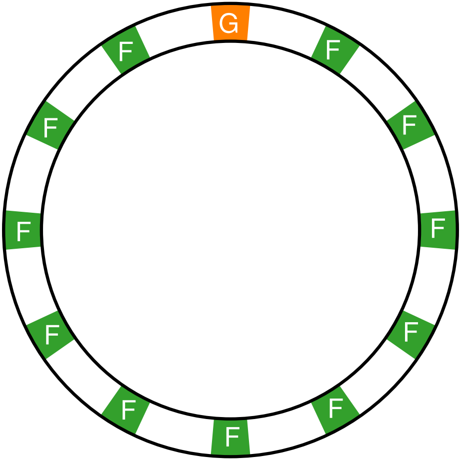
$$\rho\left(\mathcal{C}4\mathcal{P}\right) = \liminf_{G \in \mathcal{C}4\mathcal{P}} \frac{\mathrm{circ}(G)}{|V(G)|} \leq \lim_{k \to \infty} \frac{38k - (k-2)}{38k} = \frac{37}{38}$$

Higher genus

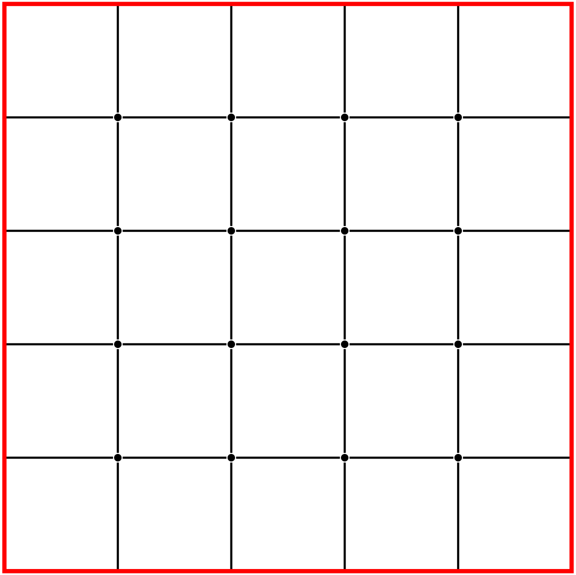
Theorem (Lo, Schmidt, VC, and Zamfirescu)

For every $g \geq 0$, the shortness coefficient of the class of cyclically 4-edge-connected cubic graphs of genus g is at most $\frac{37}{38}$.

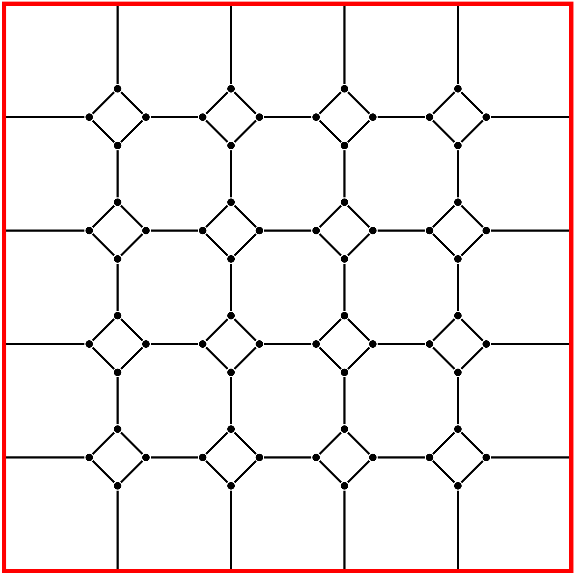
Increasing the genus



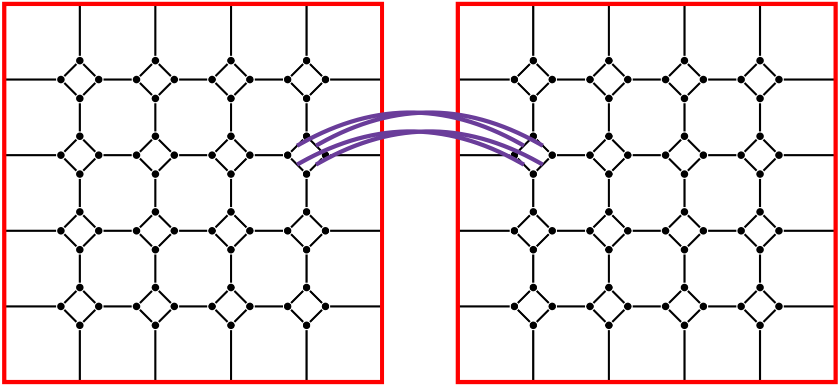
A fragment with arbitrary genus



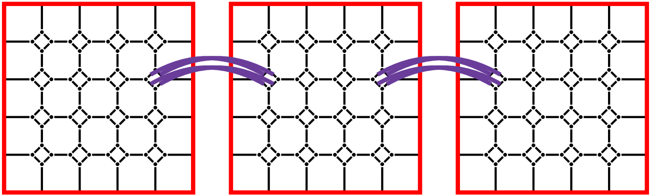
A fragment with arbitrary genus



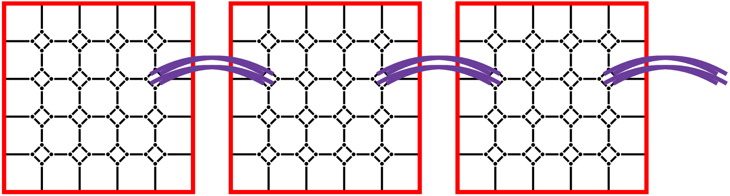
A fragment with arbitrary genus



A fragment with arbitrary genus



A fragment with arbitrary genus

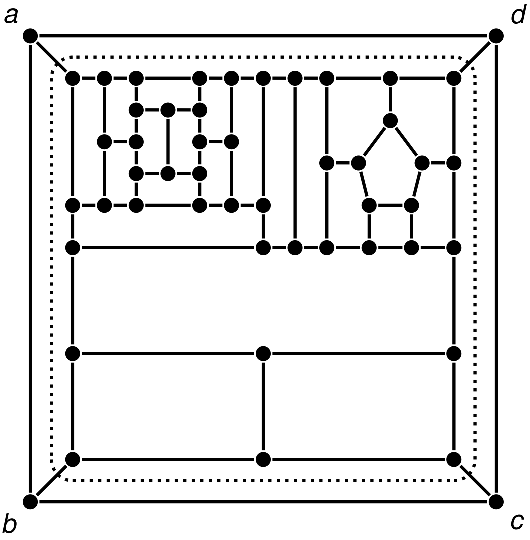


Bounded face length

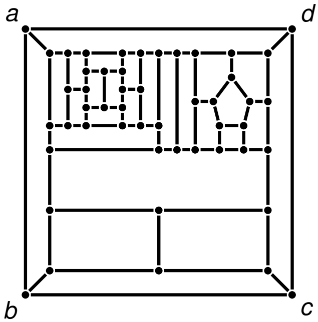
Theorem (Lo, Schmidt, VC, and Zamfirescu)

For all $\ell \geq 23$, the shortness coefficient of the class of cyclically 4-edge-connected cubic plane graphs with faces of length at most ℓ is at most $\frac{45}{46}$.

A second fragment

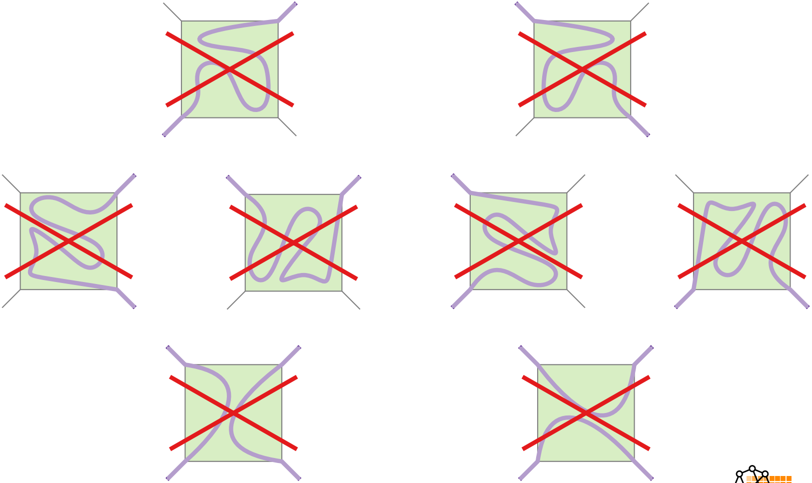


A second fragment



- H is not hamiltonian
- $H - a$ is not hamiltonian
- $H - d$ is not hamiltonian

A second fragment

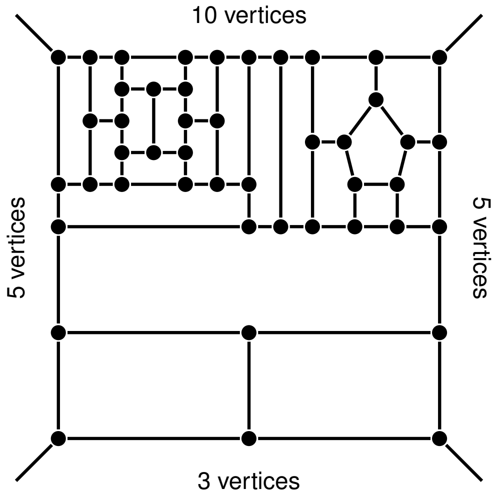


A new bound

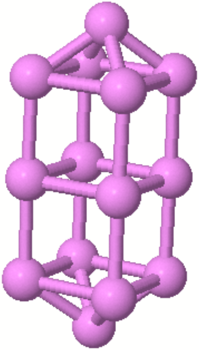
Replacing each vertex of a 4-connected 4-regular planar graph on k vertices by this fragment results in a cyclically 4-edge-connected cubic planar graph in which each cycle spanning multiple fragments misses at least one vertex in each fragment.

$$\rho\left(\mathcal{C}4\mathcal{P}\right) = \liminf_{G \in \mathcal{C}4\mathcal{P}} \frac{\operatorname{circ}(G)}{|V(G)|} \leq \lim_{k \to \infty} \frac{45k}{46k} = \frac{45}{46}$$

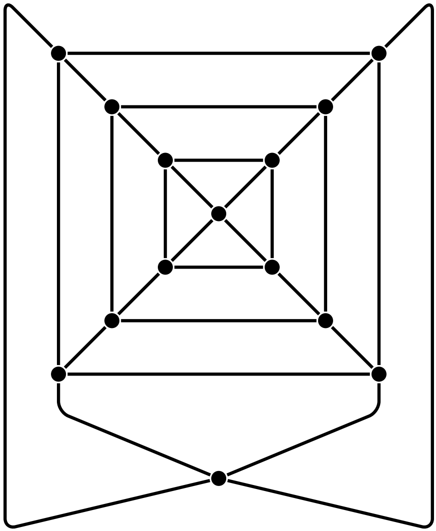
Bounded face length



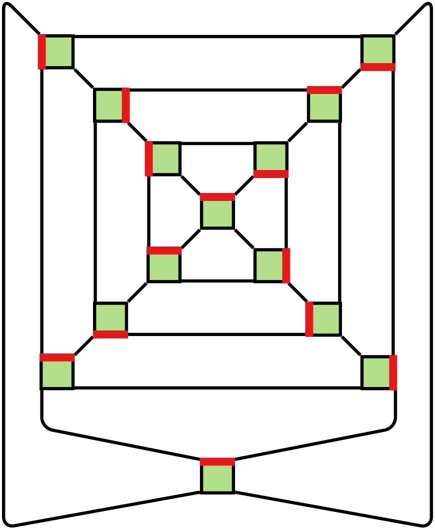
Bounded face length



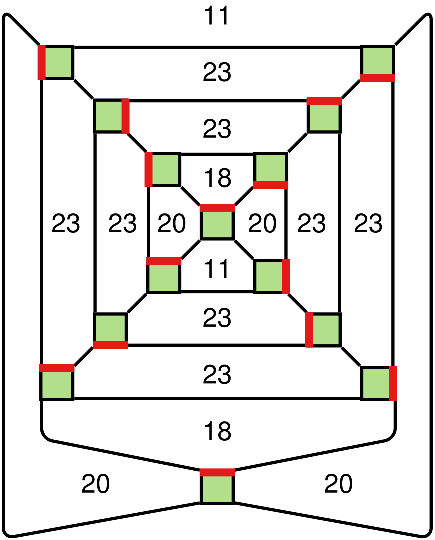
Bounded face length



Bounded face length



Bounded face length

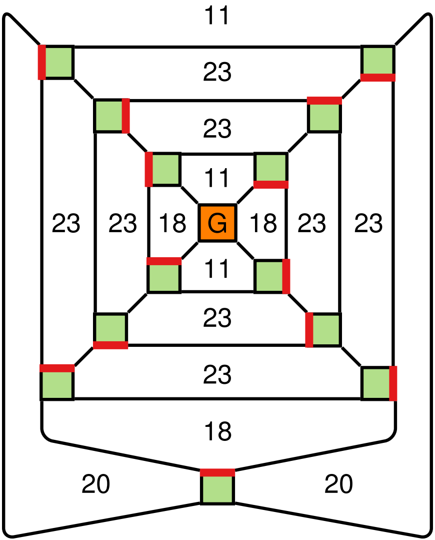


Increasing the genus

Theorem (Lo, Schmidt, VC, and Zamfirescu)

For every $g \geq 0$ and for every $\ell \geq 23$, the shortness coefficient of the class of cyclically 4-edge-connected cubic graphs of genus g with faces of length at most ℓ is at most $\frac{45}{46}$.

Increasing the genus



General cubic graphs

Theorem (Lo, Schmidt, VC, and Zamfirescu)

Let G be a cyclically 4-edge-connected cubic graph on n vertices.

Then $\rho(\mathcal{C}4) \leq \dfrac{circ(G)-2}{n-2}$, and if there exist adjacent vertices v, w in G such that $G - v - w$ is planar, then $\rho(\mathcal{C}4\mathcal{P}) \leq \dfrac{circ(G)-2}{n-2}$.

Corollary

$\rho(\mathcal{C}4) \leq \frac{7}{8}$ and $\rho(\mathcal{C}4\mathcal{P}) \leq \frac{39}{40}$.

Ongoing/future work

- $\frac{3}{4} \leq \rho(\mathcal{C}4\mathcal{P}) \leq \frac{37}{38}$
  - shrink the gap
  - fragments are smallest possible
  - missing more vertices
- shortness exponent of quartic/quintic polyhedra and polyhedra with two types of degrees